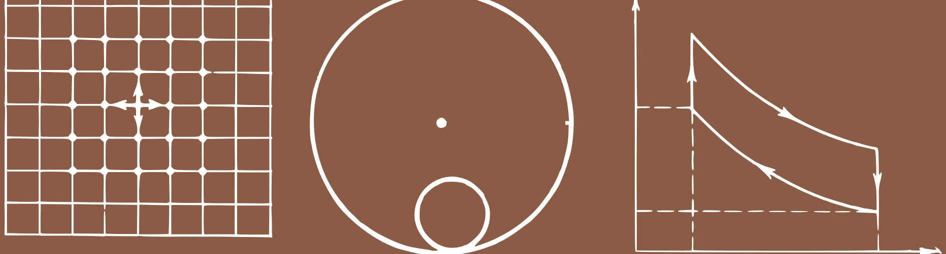
Problems in Theoretical Physics



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Problems in Theoretical Physics

ABOUT THE BOOK

This book is a collection
of problems covering mechanics,
electrodynamics, nonrelativistic
quantum mechanics,
statistical physics
and thermodynamics.
Each section opens
with a brief outline
of the main laws
and relationships used
to solve the problems.
Also information
about the needed
mathematical apparatus is included.
Along with answers there
are guides to solving
the more complicated problems.
SI units are used
throughout the book.
Problems in Theoretical Physics
is intended
for physics majors
at universities
and other institutions
of higher learning.
Some of the problems
are specifically
for students majoring
in theoretical physics.
Certain ones can be used
in the physics
and mathematics departments
of teachers colleges.

ИЗДАТЕЛЬСТВО · МИР ·

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Problems in Theoretical Physics

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TO THE READER

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На английском языке

Preface

This text incorporates problems which have been used for several years at seminars in courses in classical mechanics, electrodynamics, quantum mechanics, and statistical physics and thermodynamics at the T. G. Shevchenko State University in Kiev.

The text draws largely on the *Course of Theoretical Physics* by L. D. Landau and E. M. Lifshitz, but also makes use of other textbooks and handbooks recommended for the university course in theoretical physics. Some of the problems have been taken from published problem books listed at the end of this book, but many are original.

The student will be able to solve the problems if he has a good knowledge of the fundamentals of theoretical physics, which are briefly outlined in each section of this book. All the problems use the International System of Units (SI).

The section on classical mechanics was compiled by A. M. Fedorchenko, on electrodynamics by V. I. Sugakov, on quantum mechanics by O. F. Tomasevich, and on statistical physics and thermodynamics by L. G. Grechko.

The Authors

Contents

SECTION I Classical Mechanics

The mechanics of systems with a finite number of degrees of freedom. In mechanics a particle is a material body of mass m whose position in space is determined by three coordinates.

The mechanical state of a system of n particles is characterized by $3n$ coordinates and the $3n$ time derivatives of these coördinates. The law involving changes in the state of a mechanical system in time is defined by Newton's equations

$$m_i \ddot{\mathbf{r}}_i = \mathbf{F}_i, \quad i = 1, 2, \ldots, n \tag{I-1}$$

where $\mathbf{F}_i$ is the resultant of all the forces acting on the ith particle; these include both internal forces (those originating in the particles of the system) and external forces (those having a source outside the system and such as considered given at any instant of time).

From the standpoint of mathematics equation (I-1) is a system of $3n$ differential equations. For this reason the basic problem of mechanics consists in finding a solution for this system. We know from the theory of differential equations that to find an unambiguous solution of the system we must indicate $6n$ values $\mathbf{r}_i^0$, $\dot{\mathbf{r}}_i^0$ at a definite instant of time. In short, the mechanical state of a system at any subsequent time is determined by its initial mechanical state $\mathbf{r}_i^0$, $\dot{\mathbf{r}}_i^0$ and by the forces acting on each particle in the system.

Equations (I-1) are valid only in inertial frames of reference. An inertial frame of reference is one in which a particle free from forces, i.e. an isolated particle, is in uniform rectilinear motion. The first law of mechanics states that such frames do exist.

Forces acting between two particles are represented by the formula

$$\mathbf{F}_{ij}=F(r_{ij})\frac{\mathbf{r}_{ij}}{r_{ij}}$$

which reflects the following properties (Fig. 1):

(1) $\mathbf{F}_{ij} = -\mathbf{F}_{ji}$;

(2) $\mathbf{F}_{ij} \parallel \mathbf{r}_{ij}$;

(3) the magnitude of the force depends only on the distance between the two particles.

Classical mechanics rests on the three laws of Newton, which were deduced from experiments and observations of mechanical motion. All other assertions and laws of mechanics, valid for specific conditions and specific models, are corollaries of these three laws.

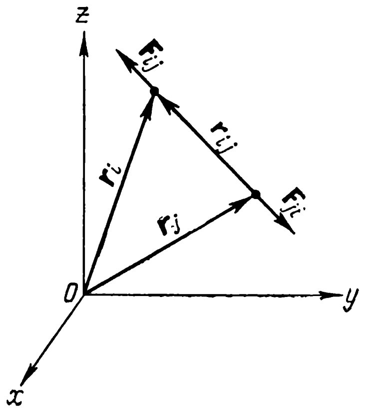

Fig. 1

In a noninertial frame of reference (one moving with acceleration) equations (I-1) do not hold. But we can preserve the form of equations (I-1) by introducing what are called forces of inertia, whose origin cannot be explained by the action of any specific particles. The forces are due to the fact that the frame of reference moves with acceleration. The equation of motion for a particle in a noninertial frame of reference is

$$m\ddot{\mathbf{r}} = \mathbf{F} + \mathbf{F}_{\text{iner}}$$

where $\mathbf{F}_{\text{iner}} = -m(\ddot{\mathbf{R}}_0 + [\dot{\boldsymbol{\omega}} \times \mathbf{r}] + [\boldsymbol{\omega} \times [\boldsymbol{\omega} \times \mathbf{r}]] + 2[\boldsymbol{\omega} \times \dot{\mathbf{r}}])$ is the force of inertia. $\ddot{\mathbf{R}}_0$ is the acceleration of the coordinate origin and $\boldsymbol{\omega}$ is the angular velocity of this frame [see formulas (I-23) and (I-24)].

If we proceed from the second law of Newton (I-1) and the first property of the forces of interaction (see above),

we can prove that the time derivative of the momentum vector of a system of particles equals the sum of all the external forces, $\mathbf{F}_{ext}$:

$$\frac{d\mathbf{p}}{dt} = \mathbf{F}_{ext} \tag{I-2}$$

where $\mathbf{p} = \sum_{i=1}^{n} m_i \dot{\mathbf{r}}_i$; n is the number of particles in the system.

If the system is closed, i.e. $\mathbf{F}_{ext}$ equals zero, equation (I-2) gives us the law of conservation of momentum:

$$\mathbf{p} = \text{constant}$$

If we introduce the notion of the centre of mass of a system

$$\mathbf{R} = \frac{\sum_{i=1}^{n} m_i \mathbf{r}_i}{M}$$

where $M = \sum_{i=1}^{n} m_i$, equation (I-2) takes the form

$$M\ddot{\mathbf{R}} = \mathbf{F}_{ext} \tag{I-3}$$

If the system is closed, it follows from equation (I-3) that

$$\dot{\mathbf{R}} = \text{constant}$$

Thus, the velocity of the centre of mass of a closed system remains constant.

From equation (I-2) we can deduce the law of motion of a body having variable mass, i.e. the law of jet propulsion. In the simplest case, if the main body (of mass m) is losing or gaining mass, the law of jet propulsion (Meshcherskii's formula) takes the following form:

$$m\frac{d\mathbf{v}}{dt} = \mathbf{F}_{ext} + \frac{dm_1}{dt}\mathbf{u}_1 - \frac{dm_2}{dt}\mathbf{u}_2 \tag{I-4}$$

where m_1 is the mass gained, $\mathbf{u}_1$ is its velocity relative to the main body, and m_2 and $\mathbf{u}_2$ are the respective values for the lost mass.

Proceeding from the second law of Newton (I-1) and the first two properties of the forces of interaction, we can

prove that the time derivative of the angular momentum of a system of particles equals the sum of the moments of all the external forces, **N**:

$$\frac{d\mathbf{L}}{dt} \equiv \dot{\mathbf{L}} = \sum_{i=1}^{n} [\mathbf{r}_i \times \mathbf{F}_i] = \mathbf{N} \tag{I-5}$$

where $\mathbf{L} = \sum_{i=1}^{n} m_i [\mathbf{r}_i \times \dot{\mathbf{r}}_i]$.

We must bear in mind that the radius vectors $\mathbf{r}_i$ of the particles in the system, which vectors enter into the definitions of the angular momentum and the moment of an external force, must issue from the same point because both depend on the choice of the coordinate origin.

Newton's third law makes it possible to introduce the concept of the potential of a force according to the formula

$$\mathbf{F}_{ij} = -\mathrm{grad}_i\, U(r_{ij}) \tag{I-6}$$

where the potential $U(r_{ij})$ depends only on the distance between the interacting particles.

We can use the potential concept to prove the following theorem on the basis of Newton's laws of motion: a change in the mechanical energy of a system equals the work done by external forces, i.e.

$$d\left(K + \frac{1}{2}\sum_{ij} U_{ij}\right) = dA \tag{I-7}$$

where by definition $K = \frac{1}{2}\sum_{i=1}^{n} m_i \dot{\mathbf{r}}_i^2$ and $dA = \sum_{i=1}^{n} (\mathbf{F}_{ext})_i\, d\mathbf{r}_i$.

The law of conservation of energy holds for closed systems:

$$E = K + \frac{1}{2}\sum_{ij} U_{ij} = \text{constant} \tag{I-8}$$

If a part of the external forces has a potential V, we can write formula (I-7) as

$$\frac{d}{dt}\left(K + \frac{1}{2}\sum_{ij} U_{ij} + V\right) = -\frac{\partial V}{\partial t} + \sum_{i=1}^{n} (\dot{\mathbf{r}}_i \cdot \mathbf{f}_i)$$

where $\mathbf{f}_i$ is a nonpotential force.

Thus, a closed mechanical system always has seven integrals of motion (seven functions of coordinates and velocities), which remain constant upon motion. In the general case the number of integrals of motion, which do not depend on time, is $2k - 1$ for a closed system, where k is the number of degrees of freedom. The seven aforementioned integrals of motion play a special role in physics. There are two main reasons for this. First, these integrals of motion always exist regardless of the number of particles in the system (for a single particle not all are independent). Second, their existence can also be proved by the fundamental properties of space-time. For instance, the law of conservation of momentum follows from the homogeneity of space (all points in space have the same status); the law of conservation of angular momentum follows from the isotropy of space (all directions in space have the same status); the law of conservation of energy follows from the homogeneity of time (all instants of time are equivalent).

The laws of motion have other forms than Newton's. Using the Lagrangian function (or, simply, the Lagrangian) and the generalized coordinates, we can write equations (I-1) in the following form:

$$\frac{d}{dt}\left(\frac{\partial \mathscr{L}}{\partial \dot{q}_i}\right)-\frac{\partial \mathscr{L}}{\partial q_i}=Q_i' \qquad \text{(I-8}'\text{)}$$

where $\mathscr{L}$ is the Lagrangian defined as $\mathscr{L} = K - V$ (K is the kinetic energy and V the potential energy of the system); q_i are the generalized coordinates, i.e. any coordinates that satisfy the sole requirement that the Cartesian coordinates (used in the system of Newtonian equations) are at any instant of time uniquely expressed in terms of all the q's:

$$\mathbf{r}_s=\mathbf{r}_s(q_1, \ldots, q_k, t)$$

$$Q_i'=\sum_{s=1}^{n}\left(\mathbf{f}_s\cdot\frac{\partial \mathbf{r}_s}{\partial q_i}\right)$$

where $\mathbf{f}_s$ is a nonpotential force; the subscript k is the number of degrees of freedom.

If there are nonpotential forces in the system but the generalized force corresponding to them can be written as

$$Q_i' = -\frac{\partial U}{\partial q_i} + \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}_i}\right)$$

where U is a function of the coordinates and velocities, the Lagrange equations of the second kind take the form

$$\frac{d}{dt}\left(\frac{\partial \mathscr{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathscr{L}}{\partial q_i} = 0 \qquad \text{(I-9)}$$

where $\mathscr{L} = K - V + U$. For example, the Lorentz force

$$\mathbf{f} = e\mathbf{E} + e\,[\dot{\mathbf{r}} \times \mathbf{B}]$$

defined by the equations

$$\mathbf{E} = -\operatorname{grad}\varphi - \frac{\partial \mathbf{A}}{\partial t}$$

$$\mathbf{B} = \operatorname{curl}\mathbf{A}$$

is a nonpotential force. It can be written as

$$f_x = eE_x + e\,(\dot{y}B_z - \dot{z}B_y) = -\frac{\partial U}{\partial x} + \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{x}}\right)$$

where

$$U = e\varphi + e\,(\dot{\mathbf{r}} \cdot \mathbf{A})$$

The Lagrange equations can be obtained from the variational principle, which states that if we introduce the functional S, called action, according to the formula

$$S = \int_{t_1}^{t_2} \mathscr{L}(t,\ q_i,\ \dot{q}_i)\,dt, \qquad \text{(I-10)}$$

the actual motion will be described by such functions $q_i(t)$ as ensure a minimum of the functional S provided that $q_i(t_1)$ and $q_i(t_2)$ are given.

The Lagrange equations are a system of k second-order differential equations. We know from mathematics that a system of k second-order differential equations can be reduced to a system of $2k$ first-order differential equations. In mechanics this is done by introducing the Hamiltonian

function (or, simply, the Hamiltonian), which is a function of the generalized coordinates and momenta. The generalized momenta are defined by the formula

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \tag{I-11}$$

Since the Lagrangian is a quadratic function of the generalized velocities, formulas (I-11) give a (linear and single-valued) relationship between the generalized velocities and the generalized momenta.

The Hamiltonian is related to the Lagrangian in the following way:

$$\mathcal{H}(p_i, q_i, t) = \sum_{i=1}^{f} p_i \dot{q}_i - \mathcal{L}(q_i, \dot{q}_i, t) \tag{I-12}$$

All the generalized velocities in the right-hand side of (I-12) must be expressed in terms of the generalized momenta according to (I-11).

The canonical equations of Hamilton are

$$\dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q_i} \tag{I-13}$$

$$\dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i} \tag{I-14}$$

Equations (I-13) and (I-14) are a system of $2k$ first-order differential equations.

In some cases the interaction of bodies is of a peculiar nature, the nature of a constraint. Constraints impose certain restrictions on changes in position or velocity. There is a fairly large class of so-called holonomic constraints, i.e. restrictions on position that can be expressed by algebraic equations:

$$f_\alpha(x_1, \ldots, x_n, t) = 0, \quad \alpha = 1, 2, \ldots, s \tag{I-15}$$

These are the equations of constraints.

To solve problems involving constraints we can use the Lagrange equations of the second kind, if we introduce such generalized coordinates as satisfy the equations of constraints automatically, or we can use the Lagrange

equations of the first kind in the following form:

$$m_i\ddot{\mathbf{r}}_i = \mathbf{F}_i + \sum_{\alpha=1}^{s} \lambda_\alpha \operatorname{grad}_i f_\alpha \tag{I-16}$$

which must be solved together with (I-15).

If we define the product $-m_i\ddot{\mathbf{r}}_i$ as the force of inertia, we can formulate the d'Alembert principle: a system moves in such a way that on any virtual displacement the work of all the forces, including forces of inertia, at any instant of time equals zero, i.e.

$$\sum_{i=1}^{n} (\mathbf{F}_i - m_i\ddot{\mathbf{r}}_i)\,\delta\mathbf{r}_i = 0 \tag{I-17}$$

In the absence of constraints this principle gives us the Newtonian equations (I-1). In the case of ideal constraints we get the Lagrange equations of the first kind.

If a system of points (particles) rests while the constraints act on it, the principle (I-17) takes the following form:

$$\sum_{i=1}^{n} (\mathbf{F}_i \cdot \delta\mathbf{r}_i) = 0$$

This equation expresses the principle of virtual displacements, which is the basis of statics. If we add to it the equations of constraints, we can find the condition for the equilibrium of a system of particles.

Solution of equations (I-1) gives us all the information about the mechanical state of a system consisting of any number of particles having an arbitrary law of interaction. However, even the three-body problem (for instance, the problem of three particles interacting via the Coulomb force) poses great mathematical difficulties. For this reason a variety of approximate methods or models that to one degree or another reflect the properties of actual system are used to solve such problems. One is the model of a rigid body. In mechanics a rigid body is a system of particles whose distances from each other remain constant in time. Such a body acts as a single whole while it is in motion.

A rigid body has six degrees of freedom, which can be chosen in the following way. Let us specify an arbitrary

point O of a rigid body in an inertial frame of reference XYZ (Fig. 2). This point will be called the pole. The pole may coincide with the centre of mass, which is defined by the formula

$$\mathbf{R}_c = \frac{\iiint \mathbf{r}\rho \, dV}{M} \qquad \text{(I-18)}$$

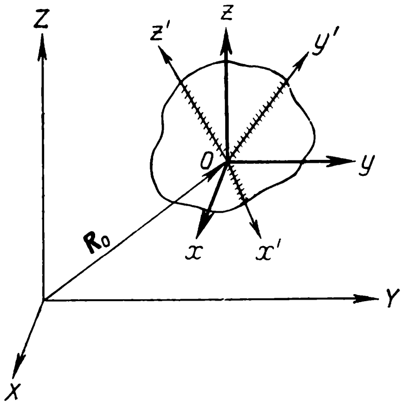

Fig. 2

The coordinates of the pole, X_0, Y_0, Z_0, are the three coordinates that describe the translational motion of a rigid body. If we fix the pole O, the body can revolve around it. This will change the orientation of a coordinate system $x'y'z'$ attached to the body. The orientation of one coordinate system in relation to another can be described by the rotation matrix α_{ij}, which links the sets of basis vectors of these systems of (Cartesian) coordinates:

$$\begin{aligned} \mathbf{i} &= \alpha_{11}\mathbf{i}' + \alpha_{12}\mathbf{j}' + \alpha_{13}\mathbf{k}' \\ \mathbf{j} &= \alpha_{21}\mathbf{i}' + \alpha_{22}\mathbf{j}' + \alpha_{23}\mathbf{k}' \\ \mathbf{k} &= \alpha_{31}\mathbf{i}' + \alpha_{32}\mathbf{j}' + \alpha_{33}\mathbf{k}' \end{aligned} \qquad \text{(I-19)}$$

The rotation matrix has the property

$$\sum_{i=1}^{3} \alpha_{ik}\alpha_{is} = \delta_{ks} \qquad \text{(I-20)}$$

The rotation matrix links the components of any vector $\mathbf{F}$ in different coordinate systems:

$$\begin{aligned} F_x &= \alpha_{11}F'_x + \alpha_{12}F'_y + \alpha_{13}F'_z \\ F_y &= \alpha_{21}F'_x + \alpha_{22}F'_y + \alpha_{23}F'_z \\ F_z &= \alpha_{31}F'_x + \alpha_{32}F'_y + \alpha_{33}F'_z \end{aligned} \qquad \text{(I-21)}$$

Since the nine matrix elements α_{ij} are restricted by six formulas of type (I-20), we can express the rotation matrix

in terms of three independent parameters. The Euler angles θ, ψ, φ (Fig. 3) are used for these three parameters. The possible values of these variables are

$$0 < \theta < \pi, \quad 0 \leqslant \psi \leqslant 2\pi, \quad 0 \leqslant \varphi \leqslant 2\pi$$

The rotation matrix is expressed in terms of the angles of rotation as follows:

$$\alpha_{ij} =$$
$$= \begin{pmatrix} (\cos\psi\cos\varphi - & -(\sin\psi\cos\theta\cos\varphi + & \sin\psi\sin\theta \\ -\sin\psi\cos\theta\sin\varphi) & +\cos\psi\sin\varphi) & \\ (\sin\psi\cos\varphi + & (\cos\psi\cos\theta\cos\varphi - & -\cos\psi\sin\theta \\ +\cos\psi\cos\theta\sin\varphi) & -\sin\psi\sin\varphi) & \\ \sin\theta\sin\varphi & \cos\varphi\sin\theta & \cos\theta \end{pmatrix} \tag{I-22}$$

Thus, the three independent coordinates X_0, Y_0, Z_0 of the pole, which characterize the translational motion of a rigid body, and the three Euler angles θ, φ, ψ, which characterize the rotational motion about pole O, form six variables that fully and uniquely determine the position of a rigid body in space. Correspondingly, the first time derivatives of the variables describe the velocity of the rigid body. But to characterize the angular velocity in the mechanics of rigid bodies we use not the derivatives $\dot\theta$, $\dot\varphi$, $\dot\psi$ but the angular velocity vector $\boldsymbol{\omega}$, which is introduced by the Poinsot formulas:

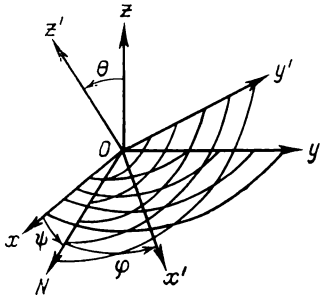

Fig. 3

$$\frac{d\mathbf{i}'}{dt} = [\boldsymbol{\omega} \times \mathbf{i}'], \quad \frac{d\mathbf{j}'}{dt} = [\boldsymbol{\omega} \times \mathbf{j}'], \quad \frac{d\mathbf{k}'}{dt} = [\boldsymbol{\omega} \times \mathbf{k}'] \tag{I-23}$$

where $\mathbf{i}'$, $\mathbf{j}'$, $\mathbf{k}'$ are the basis vectors of the coordinate system that is attached to the body. We can obtain the components of $\boldsymbol{\omega}$ in terms of the Euler angles and their time

derivatives:

$$\begin{aligned}
\omega_{x'} &= \dot{\theta}\cos\varphi + \dot{\psi}\sin\theta\sin\varphi \\
\omega_{y'} &= -\dot{\theta}\sin\varphi + \dot{\psi}\sin\theta\cos\varphi \qquad \text{(I-24)} \\
\omega_{z'} &= \dot{\varphi} + \dot{\psi}\cos\theta
\end{aligned}$$

These are the kinematic equations of Euler.

Using the definition (I-2) of momentum for a system of particles, we obtain the formula for the momentum of a rigid body

$$\mathbf{P} = M\,(\dot{\mathbf{R}}_0 + [\boldsymbol{\omega} \times \mathbf{R}_c]) \qquad \text{(I-25)}$$

where M is the body's mass and $\mathbf{R}_c$ can be found by (I-18).

If we proceed from the definition (I-5) of angular momentum for a system of particles, we obtain the formula for the angular momentum of a rigid body about the origin of an inertial frame of reference XYZ:

$$\mathbf{L} = M\,[\mathbf{R}_0 \times \dot{\mathbf{R}}_0] + M\,[\mathbf{R}_c \times \dot{\mathbf{R}}_0] + M\,[\mathbf{R}_0 \times [\boldsymbol{\omega} \times \mathbf{R}_c]] + \mathbf{L}_{rot} \qquad \text{(I-26)}$$

where $\mathbf{R}_c$ is the centre-of-mass vector in relation to the pole ($\mathbf{R}_c = 0$ if the centre of mass is selected for the pole). The last member in (I-26) is the angular momentum about the pole O.

If we introduce the inertia tensor, which is defined by the integrals

$$\begin{aligned}
I_{xx} &= \iiint (y^2+z^2)\,\rho\,dV, & I_{xy} &= I_{yx} = -\iiint xy\rho\,dV \\
I_{yy} &= \iiint (x^2+z^2)\,\rho\,dV, & I_{yz} &= I_{zy} = -\iiint yz\rho\,dV \qquad \text{(I-27)} \\
I_{zz} &= \iiint (x^2+y^2)\,\rho\,dV, & I_{zx} &= I_{xz} = -\iiint zx\rho\,dV
\end{aligned}$$

the components of $\mathbf{L}_{rot}$ will be expressed in terms of the components of $\boldsymbol{\omega}$ via the inertia tensor:

$$(\mathbf{L}_{rot})_i = \sum_{j=1}^{3} I_{ij}\omega_j \qquad \text{(I-28)}$$

Proceeding from the definition (I-7) of kinetic energy for a system of particles, we obtain the formula for the

kinetic energy of a rigid body:

$$K = \frac{1}{2} M\dot{\mathbf{R}}_0^2 + M\dot{\mathbf{R}}_0 [\boldsymbol{\omega} \times \mathbf{R}_c] + \frac{1}{2} (\boldsymbol{\omega} \cdot \mathbf{L}_{\text{rot}}) \qquad \text{(I-29)}$$

We can always select a system of coordinates (attached to the body and with an origin chosen arbitrarily) in which the symmetric inertia tensor will have none but diagonal components. Such a system is called the set of principal axes. In this system formulas (I-28) become much simpler:

$$L_1 = I_1\omega_1, \quad L_2 = I_2\omega_2, \quad L_3 = I_3\omega_3 \qquad \text{(I-30)}$$

The formula for the kinetic energy of rotation is also simplified:

$$T_{\text{rot}} = \frac{1}{2} (\boldsymbol{\omega} \cdot \mathbf{L}_{\text{rot}}) = \frac{1}{2} (I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2) \qquad \text{(I-31)}$$

We can obtain the motion equations for a rigid body by using the corollaries (I-2) and (I-5) of the Newtonian law. When the centre of mass is taken for the pole, these equations become

$$\left.\begin{aligned}
M\left(\frac{d^2X_0}{dt^2} + \omega_2\dot{Z}_0 - \omega_3\dot{Y}_0\right) &= F_1 \\
M\left(\frac{d^2Y_0}{dt^2} + \omega_3\dot{X}_0 - \omega_1\dot{Z}_0\right) &= F_2 \\
M\left(\frac{d^2Z_0}{dt^2} + \omega_1\dot{Y}_0 - \omega_2\dot{X}_0\right) &= F_3 \\
I_1\frac{d\omega_1}{dt} + (I_3 - I_2)\,\omega_2\omega_3 &= N_1 \\
I_2\frac{d\omega_2}{dt} + (I_1 - I_3)\,\omega_1\omega_3 &= N_2 \\
I_3\frac{d\omega_3}{dt} + (I_2 - I_1)\,\omega_1\omega_2 &= N_3
\end{aligned}\right\} \qquad \text{(I-32)}$$

where the subscripts 1, 2, 3 designate the components of vectors along the principal axes, and X_0, Y_0, Z_0 are the coordinates of the centre of mass in the same coordinate system. The moments of forces that enter into the total moment of forces $\mathbf{N}$ are taken relative to the centre of mass (pole).

The system of six equations (I-32) is a complete system of differential equations for finding the six functions $X(t)$,

$Y(t)$, $Z(t)$, $\varphi(t)$, $\theta(t)$, $\psi(t)$. As we see from this system, the character of motion of a rigid body depends not only on the mass but also on its distribution, since the moments of inertia I_1, I_2, I_3 depend on the body's shape and the distribution of its density.

Another way to formulate the motion equations of a rigid body is to construct the Lagrangian. The kinetic energy is expressed by formula (I-29). Given the forces, we can find the corresponding potentials and construct the Lagrangian. Then we formulate the Lagrange equations of the second kind.

The mechanics of continuous media. When we examined the motion of a rigid body, we assumed that the distances between the particles constituting the body remain unchanged. But we know from experience that solids have a fairly wide range of mechanical states (deformation, the propagation of sound) that cannot be described by the model of a rigid body. Thus we must consider the internal motion of the particles of a solid in relation to each other.

A similar situation appears when we examine the motion of liquids and gases (fluids). The main difference between the motion of fluids and the motion of elastic solids is that the particles constituting the liquid or gas can move a considerable distance from their initial position. In other words, in contrast to elastic solids liquids and gases have the property of fluidity. Liquids, gases and elastic solids are often designated as continuous media.

In studying the motion of liquids, gases and solids we can do one of two things. We can describe them as a system consisting of a large number of particles by assigning coordinates to each particle. Or we can consider them as continuous media, i.e. media with a continuous rather than a discrete distribution of particles. This is the approach in introducing the concept of density.

One of the main concepts in the mechanics of continuous media is the vector field of velocities. This field is the vector function of coordinates and time, $\mathbf{v}(x, y, z, t)$, that indicates the velocity of those particles of the medium that at time t pass through the point in space with coordinates x, y, z. Thus the velocity $\mathbf{v}(x, y, z, t)$ does not refer to any particle in particular but characterizes the motion of the medium as a whole.

The equations of motion for a continuous medium are obtained by applying the law of change of momentum in time (I-2) to an arbitrary volume of the medium. Forces of two kinds act on this volume:

(1) the surface forces exerted by the other parts of the medium and defined by the formula

$$(F_{\text{surface}})_i = \int p_{ij} n_j \, dS = \int \frac{\partial p_{ij}}{\partial x_j} \, dV \tag{I-33}$$

where p_{ij} is the stress tensor and n_j are the components of the outward normal to the surface of the chosen volume;

(2) the body forces (e.g., the force of gravity), defined by the space integral

$$(F_{\text{body}})_i = \int f_i \rho \, dV \tag{I-34}$$

where ρ is the density of the medium and f_i is the body force per unit mass.

The motion equation for a continuous medium has the form

$$\rho \left[\frac{\partial v_i}{\partial t} + (\mathbf{v} \cdot \nabla) v_i \right] = \frac{\partial p_{ij}}{\partial x_j} + \rho f_i \tag{I-35}$$

For the law of change of angular momentum in time to be valid, it is necessary and sufficient for the stress tensor to be symmetric:

$$p_{ij} = p_{ji}$$

From the law of conservation of mass we can obtain the relation between the field of velocities and density:

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \rho \mathbf{v} = 0 \tag{I-36}$$

This is the equation of continuity.

If the continuous medium is incompressible, the equation of continuity gives us the condition for incompressibility

$$\operatorname{div} \mathbf{v} = 0 \tag{I-37}$$

which is often used in solving problems involving the flow of liquid and even gas.

For the system of equations to be complete, we adjoin the second law of thermodynamics for continuous media

$$\rho \frac{d\varepsilon}{dt} = -\operatorname{div} \mathbf{j} + p_{ij} \frac{\partial v_i}{\partial x_j} \tag{I-38}$$

where ε is the internal energy per unit volume and $\mathbf{j} = -\varkappa \operatorname{grad} T$ is the heat flux vector.

To solve equation (I-35) we must know the stress tensor p_{ij}. For viscous fluids we can write it as

$$p_{ij} = -p\delta_{ij} + (\eta - \zeta)\,\delta_{ij} \operatorname{div} \mathbf{v} + \eta \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) \qquad \text{(I-39)}$$

Substituting this expression into (I-35), we get the Navier-Stokes equation

$$\rho \left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\, \mathbf{v} \right] = \rho \mathbf{f} - \operatorname{grad} p + \eta \Delta \mathbf{v} + \zeta \operatorname{grad} \operatorname{div} \mathbf{v} \qquad \text{(I-40)}$$

where η is the coefficient of shear viscosity and ζ the coefficient of bulk viscosity. For incompressible liquids the system of hydrodynamic equations takes the form

$$\left. \begin{aligned} \rho \left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\, \mathbf{v} \right] = \rho \mathbf{f} - \operatorname{grad} p + \eta \Delta \mathbf{v} \\ \operatorname{div} \mathbf{v} = 0 \end{aligned} \right\} \qquad \text{(I-41)}$$

These are partial differential equations. For this reason to solve system (I-41) we must fix the initial and boundary conditions. For boundary conditions we use the "sticking" condition ($\mathbf{v}_b = 0$) on stationary walls.

Equation (I-38) is not connected with system (I-41) for incompressible liquids. It is used to find the temperature as a function of coordinates and time (see Problems 109 and 111).

The equation of motion (I-35) for a continuous medium also gives us the motion equation for an elastically deformed solid. What characterizes the elastically deformed solid is that even when a fairly strong external force acts upon it, the particles constituting the solid do not move far from the equilibrium configuration, and when the force is removed, the particles return to their initial position. Thus

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\, \mathbf{v} \approx \frac{\partial^2 \mathbf{u}}{\partial t^2} \qquad \text{(I-42)}$$

where $\mathbf{u}\,(x, y, z, t)$ is the displacement vector characterizing the displacement of the parts of an elastically deformed solid from the equilibrium position.

Small elastic deformations are characterized by the symmetric strain tensor

$$u_{ij}=\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right) \tag{I-43}$$

The stress tensor and the strain tensor are related by Hooke's law

$$p_{ij}=\sum_{k,\,l=1}^{3}\lambda_{ijkl}u_{kl} \tag{I-44}$$

where λ_{ijkl} is the fourth-order tensor of elastic constants, which satisfies the symmetry relations

$$\begin{aligned}\lambda_{ijkl}&=\lambda_{jikl}\\ \lambda_{ijkl}&=\lambda_{ijlk}\\ \lambda_{ijkl}&=\lambda_{klij}\end{aligned} \tag{I-45}$$

Because of the symmetry relations (I-45) this tensor in the general case has only 21 independent variables. Thus we can rewrite Hooke's law as follows:

$$p_i=c_{ij}s_j, \qquad s_j=\frac{\partial u_i}{\partial x_j} \tag{I-46}$$

where the lower index i assumes six values:

one-index numbering	1	2	3	4	5	6
two-index notation	xx	yy	zz	yz	xz	xy

In this manner $\lambda_{xxxx}=c_{11}$, $\lambda_{xxyy}=c_{12}$, $\lambda_{xyxy}=c_{66}$, etc.

As a consequence of spatial symmetry the number of independent elastic constants can be further reduced. There are three independent elastic constants for a crystal with cubic symmetry, for instance, and matrix c_{ij} has the form

$$\begin{pmatrix} c_{11} & c_{12} & c_{12} & 0 & 0 & 0\\ c_{12} & c_{11} & c_{12} & 0 & 0 & 0\\ c_{12} & c_{12} & c_{11} & 0 & 0 & 0\\ 0 & 0 & 0 & c_{44} & 0 & 0\\ 0 & 0 & 0 & 0 & c_{44} & 0\\ 0 & 0 & 0 & 0 & 0 & c_{44} \end{pmatrix}$$

Isotropic solids are characterized by two elastic constants, and we can write Hooke's law in the following way:

$$p_{ij} = \lambda \delta_{ij} \operatorname{div} \mathbf{u} + 2\mu u_{ij} \tag{I-47}$$

The general equation of motion and equilibrium for an elastically deformed solid has the form

$$\rho \frac{\partial^2 u_i}{\partial t^2} = \frac{\partial p_{ij}}{\partial x_j} + \rho f_i \tag{I-48}$$

where p_{ij} is defined by (I-44).

Sound-energy flux per unit area (sound intensity) can be determined by the formula

$$S_i = -p_{ij} u_j$$

In isotropic media the time average of the magnitude of vector $\mathbf{S}$ for progressive monochromatic waves is

$$\overline{S} = \frac{1}{2} \rho\, v_s \omega^2 u_0^2 \tag{I-49}$$

where v_s is the velocity of longitudinal or of transverse sound waves, ρ the density of the medium, ω the frequency and u_0 the sound-wave amplitude.

Formula (I-48) incorporating (I-47) assumes for isotropic materials the form

$$\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} = \rho \mathbf{f} + (\lambda + \mu) \operatorname{grad} \operatorname{div} \mathbf{u} + \mu \Delta \mathbf{u} \tag{I-50}$$

In the theory of elasticity the boundary conditions are

$$(\mathbf{u})_{\mathrm{I}} = (\mathbf{u})_{\mathrm{II}}, \quad (p_{ij} n_j)_{\mathrm{I}} = (p_{ij} n_j)_{\mathrm{II}} \tag{I-51}$$

where n_j are the components of the outward normal to the interface of media I and II. The first condition points to the continuity of the media and the second to the properties of the forces.

PROBLEMS

1. A particle of mass m and charge e enters a homogeneous and stationary electric field $\mathbf{E}$ with a velocity v_0 perpendicular to the direction of the field. Calculate the particle's path.

2. A particle of mass m and charge e enters a homogeneous retarding electric field $\mathbf{E}$ with a velocity v_0 parallel to the direction of the field. How much time will it take the particle to return to its initial position?

3. A particle of mass m and charge e enters a homogeneous and stationary magnetic field $\mathbf{H}$ with a velocity v_0 perpendicular to the direction of the field. Calculate the particle's path.

4. Consider a homogeneous electric field that changes according to the law $\mathbf{E} = \mathbf{E}_0 \cos \omega t$. A particle of mass m

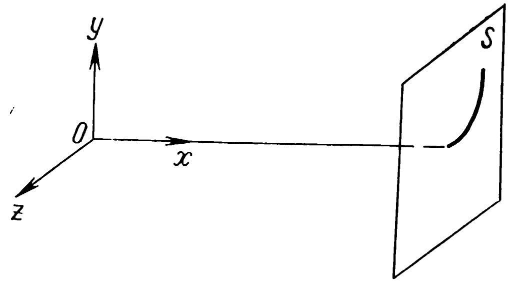

Fig. 4

and charge e enters the field with a velocity v_0 perpendicular to the direction of the field. Calculate the particle's path.

5. Consider homogeneous and stationary electric and magnetic fields $\mathbf{E}$ and $\mathbf{H}$ in a certain area of space. The angle between the two is equal to α. A particle of mass m and charge e enters this area with a velocity $\mathbf{v}_0$. Calculate the particle's path.

6. A cluster of charged particles flies out of point O, the particles having different initial velocities v_0 but directed along the x-axis (Fig. 4). Moving through a region with an electric and a magnetic field directed along the y-axis, the particles hit a fluorescent screen S, which is at a distance l from point O. Show that the track left by the particles on the screen is a parabola provided that all the velocities satisfy the condition $\omega_H l/v_0 \ll 1$.

7. A homogeneous and stationary magnetic field directed along the z-axis is created in the region $0 < y < l$, $-\infty <$

$< x < +\infty$, $-\infty < z < +\infty$ (a "magnetic wall"). A particle of mass m and charge e enters the field with a velocity $\mathbf{v}_0$ directed at an angle α to the xz-plane. The angle between the z-axis and the projection of $\mathbf{v}_0$ on the xz-plane is β (Fig. 5). Find:

(1) the condition for the particle's penetration of the magnetic wall;

(2) the direction the particle will take after penetrating the wall;

(3) the direction of the particle's reflection from the wall and determine the conditions in which the laws governing the reflection will coincide with optical laws of reflection.

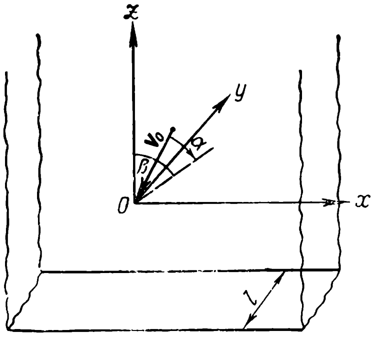

Fig. 5

8. A beam of electrons enters the space between two pairs of deflecting plates, which have the following voltages: $U_x = U_1 \sin \omega t$ on the vertical plates and $U_y = U_2 \cos \omega t$ on the horizontal ones. Find the path of the beam on the screen S (Fig. 6) if all the electrons before entering have the initial velocity v_0 parallel to the plates. The length of the plates is l and their distance from the screen is also l.

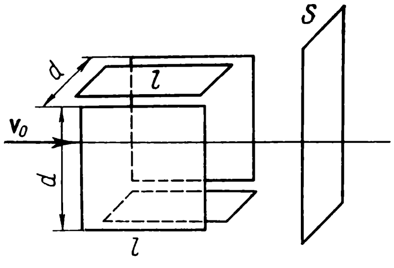

Fig. 6

9. Consider a harmonic oscillator with an electric charge e in a homogeneous and stationary magnetic field $\mathbf{H}$ (the classical Zeeman effect). Construct the motion equation of this oscillator and solve it.

10. Use the conditions of Problem 1 and also assume the particle is acted upon by a force of resistance proportional to the first power of the velocity, $\mathbf{R} = -\gamma \mathbf{v}$. Construct the motion equation and solve it.

11. Use the conditions of Problem 3 and also assume the particle is acted upon by a force of resistance proportional to the first power of the velocity, $\mathbf{R} = -\gamma\mathbf{v}$. Construct the motion equation and solve it.

12. A mass M with no initial velocity falls from a height H onto a helical spring (Fig. 7). The mass forces the spring

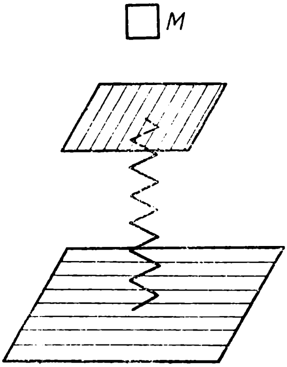

Fig. 7

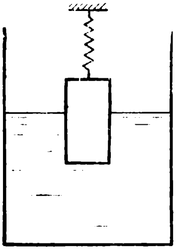

Fig. 8

to contract by h. What is the time of contraction if we neglect the action of forces of resistance and consider the mass of the spring to be a negligible quantity?

13. A mass m falls in the air with no initial velocity. Assuming the force of air resistance to be proportional to the second power of the velocity, $R = \gamma v^2$, find the velocity and position of the mass as functions of time. To what limit does the velocity tend with the passage of time?

14. A cylinder of mass M, radius r, and height h, suspended by a spring whose upper end is fixed, is submerged in water (Fig. 8). In equilibrium the cylinder sinks to 1/2 of its height. At a certain moment the cylinder was submerged to 2/3 of its height and then with no initial velocity started to move vertically. Find the motion equation of the cylinder in relation to the position of equilibrium if the stiffness coefficient of the spring is c and the density of the water is ρ.

15. A body falls to the ground from a great height h. Neglecting the resistance of the air, find the time T that

it will take the body to reach the ground and the velocity v that the body will develop in that time. The earth's radius is R.

16. A body of mass m, thrown at an angle α to the horizontal plane with an initial velocity v_0, moves under the action of the force of gravity and the force of resistance of the air $\mathbf{R}$. The force of resistance is proportional to the first power of the velocity: $\mathbf{R} = -\gamma\mathbf{v}$. Solve the motion equation and determine the maximum height h and the

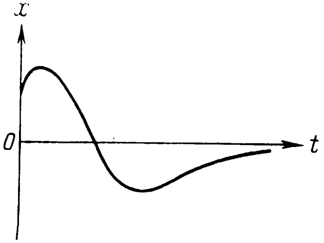

Fig. 9

distance s to the base of the maximum height along the horizontal plane.

17. A particle of mass m moves according to the law $x = a \cos \omega t$, $y = b \sin \omega t$. Determine the force that acts on the particle at every point of the path.

18. A particle of mass m moves along the path

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

with an acceleration parallel to the y-axis. At $t = 0$ the particle is at the point $x = 0$, $y = b$ and has a velocity v_0. Determine the force that acts on the particle at every point of the path.

19. Find the equation of harmonic oscillations of a particle that is acted upon by a driving force $f = f_0 e^{-\alpha t} \cos \omega t$. The force of friction is proportional to the first power of the velocity.

20. Consider a particle of mass m that is acted upon by an elastic force $m\omega_0^2\mathbf{r}$ and a force of friction $-m\gamma\dot{\mathbf{r}}$. Under what conditions will the motion of the particle be represented by the diagram in Fig. 9 (i.e. the particle tends to

its position of equilibrium, first passing through it)? Prove that there is only one maximum deviation from equilibrium other than the initial one.

21. A beam of electrons falls on the middle of a vane of the Crookes radiometer (Fig. 10). Determine the law that governs the motion of the vane wheel, which consists of

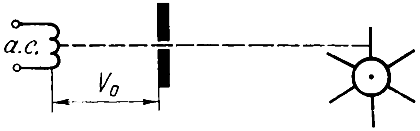

Fig. 10

six vanes, if the electrons are accelerated by a potential V_0 and the electron current is I_0. The radius of the vane wheel is R, its width is l, the thickness of each vane is d, and the

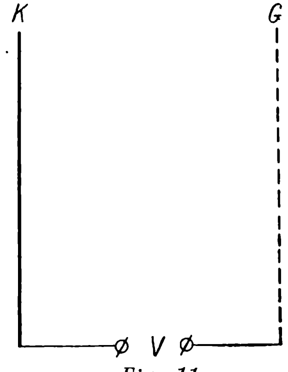

Fig. 11

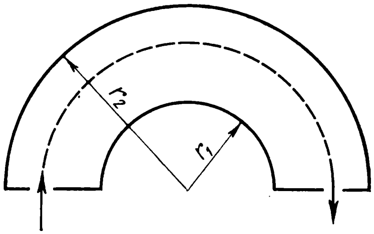

Fig. 12

density of the vane material is ρ. We ignore the friction in the bearings, the reflection of the electrons, and the secondary emission.

22. Plane K is a source of charged particles. The particles are accelerated by a potential difference V between plane K and grid G (Fig. 11) and then fly through G into space.

(1) What is the reaction force applied to the system (plane K and grid G) if we take account of the space charge?

(2) What is the power requirement for accelerating the particles?

(3) Analyze the dependence of these two quantities on the type of particles used.

23. Consider a cylindrical capacitor (Fig. 12) with a homogeneous and stationary magnetic field **H** directed along its axis (perpendicular to the paper). A particle of mass M and charge e flies into the entrance slit with a kinetic energy K. What must the potential difference between the inner and the outer cylinder be if we want the particle to fly along the capacitor's midline? How can such a device be used as a mass-analyzer for ion separation?

Hint. The electric field in a cylindrical capacitor is

$$E_r = \frac{V}{r \ln \frac{r_2}{r_1}}$$

where r_1 is the radius of the inner cylinder, r_2 the radius of the outer cylinder, and V the potential difference between the two.

24. What masses should single-charged ions have in order to fly through the device described in Problem 23 if (1) $V = 300$ V, $r_1 = 6$ cm, $r_2 = 5.4$ cm, $K = 1000$ eV, and H changes from 0 to 10 000 Oe; (2) $H = 5000$ Oe, $r_1 = 6$ cm, $r_2 = 5.4$ cm, $K = 1000$ eV, and V changes from 0 to 20 000 V?

25. The relative motion of two particles interacting via the Coulomb law $\left(V = \frac{\alpha}{r}\right)$ is represented by a conical section with the parameters

$$p = \frac{L^2}{|\mu|\alpha|} \quad \text{and} \quad e = \sqrt{1 + \frac{2EL^2}{\mu\alpha^2}}$$

where L is the relative angular momentum, μ the reduced mass, and E the energy of relative motion. Prove that even if E is negative, it always has a lower bound so that the condition $e^2 > 0$ always holds.

26. Express the period of revolution of an earth satellite in terms of various parameters.

27. Find the path of a particle of mass m moving in an external field with a potential $V = \frac{\alpha}{r} + \frac{\beta}{r^2}$. Determine the condition in which the particle will (1) "fall" on the

centre; (2) "pass" to infinity (scatter); (3) be in alternating motion.

28. Determine the relation between the impact parameter s and the scattering angle Φ using the conditions of Problem 27.

29. A particle of mass m and energy E is in a state of one-dimensional motion in a potential field $U(x)$ (Fig. 13). Determine the particle's period of motion.

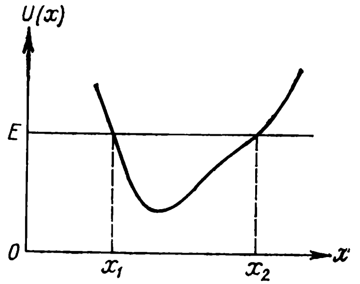

Fig. 13

30. Determine the period of one-dimensional motion of a particle of mass m and energy E in a potential field

$$U = -\frac{U_0}{\cosh^2 \alpha x}$$

$$-U_0 < E < 0.$$

31. Determine the period of one-dimensional motion of a particle of mass m and energy E in a potential field

$$U = U_0 \tan^2 \alpha x.$$

32. Find the path of a particle of mass m and energy E moving in a potential field $U = \alpha/r^2$ $(\alpha > 0)$.

33. Find the scattering angle and the effective cross section when a particle of energy E is scattered by a potential field $U = \frac{\alpha}{r^2} (\alpha > 0)$.

34. Show that when two particles interact via the Coulomb law, there is an integral of motion equal to (a vector quantity)

$$[\mathbf{v} \times \mathbf{L}] + \frac{\alpha \mathbf{r}}{r}$$

where $\mathbf{v}$ is the relative velocity, $\mathbf{r}$ the relative radius vector, $\mathbf{L} = \mu [\mathbf{r} \times \mathbf{v}]$ the relative angular momentum, and α the constant in the Coulomb law.

35. Consider a rocket on which none but a reaction force is acting. Find the relationship (Tsiolkovsky's formula) between time and the rocket's velocity if we know the law

of change of the rocket's mass $m(t)$ and the exhaust velocity u_1 relative to the rocket.

36. Orbital velocity is the minimum velocity which a body must attain to establish a permanent orbit, i.e. to become a satellite. Escape velocity is the minimum velocity which a body must attain to escape from the gravitational pull of a heavenly body. Find the orbital (v_1) and escape (v_2) velocities for the earth and the moon.

37. Determine the final amplitude (as $t \to \infty$) of the harmonic oscillations of a mass m after the action of the following force (we disregard friction):

(1) $F = F_0 \frac{t}{T}$ when $0 < t < T$, $F = F_0$ when $T < t < \infty$;

(2) $F = F_0 \frac{t}{T}$ when $0 < t < T$, $F = 0$ when $T < t < \infty$;

(3) $F = F_0$ when $0 < t < T$, $F = 0$ when $T < t < \infty$;

(4) $F = F_0 \sin \omega t$ when $0 < t < T = \frac{2\pi}{\omega}$, $F = 0$ when $T < t < \infty$. Before the force acted, the oscillator was in equilibrium.

38. A particle of mass m_1, having a velocity $\mathbf{v}_1$, is scattered by a particle of mass m_2 at an angle θ. Find the scattering angle Φ in the centre-of-mass frame, the energy transfer, and the mass ratio for which the energy transfer is maximal.

39. A particle of mass m is moving under the action of an external force $\mathbf{F} = km\mathbf{r}$, where $\mathbf{r}$ is the particle's radius vector. Determine the path of the particle if its initial position is $\mathbf{r}_0$ and its initial velocity $\mathbf{v}_0$ is perpendicular to $\mathbf{r}_0$.

40. Solve the motion equation for the cylinder of Problem 14 if the water resistance is proportional to the first power of the velocity: $\mathbf{F}_{res} = -\alpha\mathbf{v}$.

41. Prove that if the linear momentum of a system of n particles is equal to zero, then the angular momentum of the system does not depend on the choice of the point relative to which it is calculated.

42. Prove that the moment of the forces applied to a system of n particles does not depend on the choice of the

point relative to which the moment is calculated if the resultant of the forces applied to the system is equal to zero.

43. An atom consists of a nucleus of mass M and n electrons of mass m each. Eliminate the motion of the centre of mass and reduce the problem to the motion of n particles. Find the Lagrangian of the system.

44. In Problem 43 the kinetic energy in the Lagrangian is not a simple sum of squares. Prove that if the kinetic energy is expressed in terms of the Jacobi coordinates, it takes the form of a simple sum of squares. The Jacobi coordinates are

$$\rho_1 = \frac{m_1}{m_1}\mathbf{r}_1 - \mathbf{r}_2 \equiv \mathbf{r}_1 - \mathbf{r}_2$$

$$\rho_2 = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2} - \mathbf{r}_3$$

$$\cdots\cdots\cdots\cdots$$

$$\rho_j = \frac{m_1\mathbf{r}_1 + \ldots + m_j\mathbf{r}_j}{m_1 + \ldots + m_j} - \mathbf{r}_{j+1}$$

$$\cdots\cdots\cdots\cdots$$

$$\rho_n = \frac{m_1\mathbf{r}_1 + \ldots + m_n\mathbf{r}_n}{m_1 + \ldots m_n} = \mathbf{R}.$$

45. Construct the Lagrangian of a dipole whose opposite charges are of masses m_1 and m_2 and which is located in a homogeneous electric field $\mathbf{E}$.

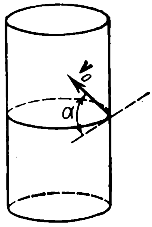

Fig. 14

46. A particle of mass m moves along the inner surface of a vertical cylinder of radius r (Fig. 14). Find the pressure that the particle exerts on the cylinder if the surface of the cylinder is considered to be perfectly smooth. The particle's initial velocity $\mathbf{v}_0$ forms an angle α with the horizontal plane.

47. In Problem 46 find the position of the particle as function of time if the particle was on the x-axis at the initial instant of time.

48. A pipe AB revolves with a constant angular velocity ω on a vertical axis CD, forming a permanent angle α with it (Fig. 15). Inside the pipe there is a ball of mass m. Determine the nature of the ball's motion if its initial velocity is equal to zero and its initial position is at a distance a from a point O. We exclude friction.

49. A thin, straight, and homogeneous rod of length l and mass M revolves with a constant angular velocity ω about a stationary point O describing a conical surface

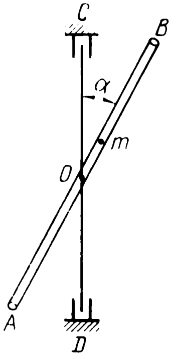

Fig. 15

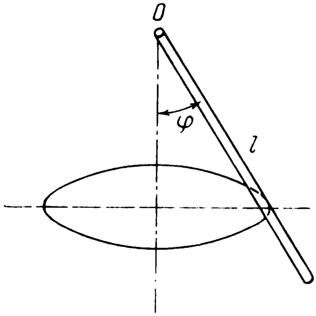

Fig. 16

(Fig. 16). Determine the angle of the rod's deviation from the vertical and the force of reaction at point O.

50. A homogeneous prism A lies on a horizontal plane. A prism B (also homogeneous) is placed on prism A (Fig. 17). The cross sections of both are right triangles, and

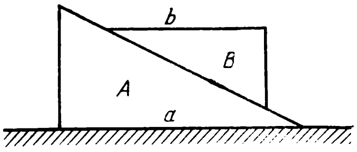

Fig. 17

the mass of prism A is n times that of B. Determine how far prism A has moved when prism B, sliding down A, reaches the horizontal plane. Assume that the prisms and the plane are perfectly smooth.

51. Two blocks of masses m_1 and m_2 connected by a massless, inextensible cord that is passed over a massless pulley A slide along the smooth sides of a rectangular wedge of mass m, which rests on a smooth horizontal plane (Fig. 18).

Find the displacement of the wedge on the horizontal plane when mass m_1 is lowered to a height h.

52. An electric motor of mass M is placed on a smooth horizontal foundation but not fastened down. A homogeneous rod of length $2l$ and mass M_1 is attached to the motor's

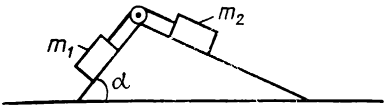

Fig. 18

shaft at a right angle. The other end of the rod has a particle of mass m attached to it (Fig. 19). The shaft turns at an angular velocity ω. Determine

(1) the equation for the horizontal motion of the motor;

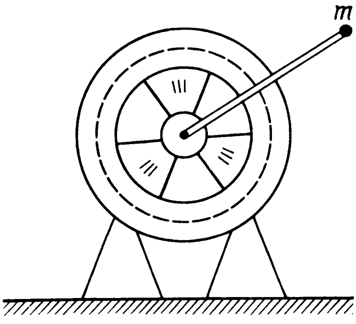

Fig. 19

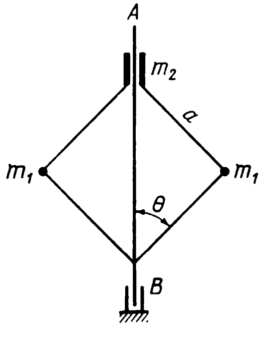

Fig. 20

(2) the maximal horizontal stress R which would act on bolts if the motor were fastened to the foundation;

(3) the required angular velocity of the shaft so that the motor will bounce on the foundation if it is not bolted down.

53. Can a complex system of particles whose centre-of-mass energy is E decay into two systems with energies E_1 and E_2?

54. Due to the rotation of the earth about its axis any freely falling body deviates from the vertical line. Find

this deviation if the initial velocity is zero. Ignore air resistance.

55. Construct the Lagrangian of a simple pendulum whose point of suspension moves in the vertical plane according to the law $y = y(t)$ and $x = x(t)$. The mass of the pendulum is m and its length is l.

56. Construct the motion equation for small oscillations of a simple pendulum whose point of suspension moves along the vertical line according to the law $x = a \cos \omega t$. The mass of the pendulum is m and its length is l.

57. A mechanical system, depicted in Fig. 20, rotates about the vertical axis AB with a constant angular velocity ω.

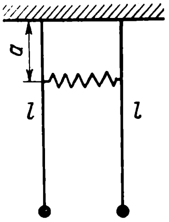

Fig. 21

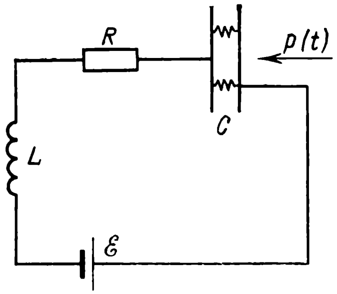

Fig. 22

The body of mass m_2 is able to move along the vertical axis. Find the Lagrangian of the system and determine the positions of equilibrium.

58. In Problem 57 determine which of the two positions of equilibrium is the position of stable equilibrium and which of unstable equilibrium.

59. Two simple pendulums of equal length l (Fig. 21) are coupled by a spring with a stiffness coefficient c at a distance a from the suspension points. Determine the frequency of small oscillations and solve the motion equation of such if at the initial instant of time one pendulum was deflected from the vertical line by an angle φ_0.

60. The capacitor microphone consists of a series-circuit of inductance L, conductance R, and capacitance C of a capacitor whose plates (one of which can move) are coupled by two springs with a general stiffness coefficient c (Fig. 22).

This circuit is connected to a cell with a constant e.m.f. $\mathscr{E}$. In the state of equilibrium $C = C_0$ and the distance between the plates is a. Construct the Lagrangian and write the Lagrange equation if the mass of the moving plate is m and a variable force $p(t)$ acts on it.

61. Find the states of equilibrium for the system of Problem 60. Determine the frequency of small oscillations.

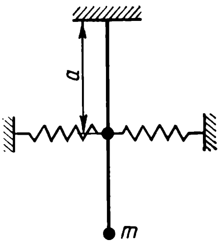

Fig. 23

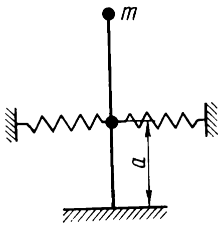

Fig. 24

62. A ball of mass m hangs at the end of a massless rigid rod of length l. Two springs with a stiffness coefficient c are connected to the rod as shown in Fig. 23. Find the frequency of small oscillations.

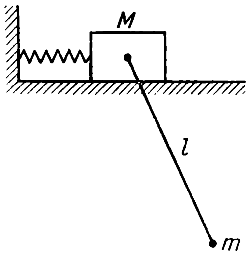

Fig. 25

63. Find the frequency of small oscillations when the pendulum of Problem 62 is turned upside down, i.e. the ball is higher than the former point of suspension (Fig. 24). Determine the state of equilibrium in this case.

64. A block of mass M, connected to a spring with a stiffness coefficient c, can move in the horizontal plane without friction. A simple pendulum of mass m and length l is fastened to it (Fig. 25). Find the Lagrangian of the system and determine the frequency of small oscillations. (The other end of the spring is fixed.)

65. Construct the Lagrangian of a simple pendulum of mass m and length l whose point of suspension moves in the horizontal plane according to the law $x = x(t)$.

66. Construct the motion equation for small oscillations of a simple pendulum of length l whose point of suspension moves on a horizontal line according to the law $x = a\cos\gamma t$.

67. A homogeneous rod BD leans against a wall (Fig. 26). The rod's lower end rests on a horizontal plane and is held

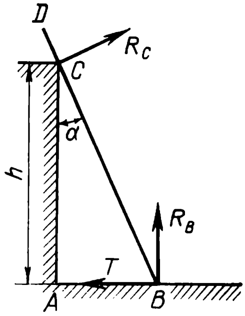

Fig. 26

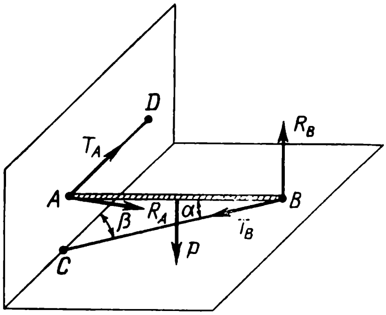

Fig. 27

by the line AB. Find the reaction forces at the points of contact and the tension in the line. The weight of the rod is P and its length is l.

68. A homogeneous rod AB leans against a vertical plane and rests on a horizontal plane (Fig. 27). Two horizontal lines AD and BC hold the rod in a fixed position; the line BC and the rod are in the same vertical plane. Find the reaction forces at the points of contact and the tension in the lines. The weight of the rod is P.

69. A ball B of mass m is suspended by a string AB and touches the smooth surface of a sphere of radius r (Fig. 28). The distance from the surface of the sphere to the static point A is d, and the length of the string is l. Find the tension in the line and the reaction force with which the sphere acts on the ball. The dimensions of the ball are negligible.

70. Two straight homogeneous rods of length a and b are rigidly fixed at a right angle whose vertex O is connected to a vertical shaft by means of a joint (Fig. 29). The shaft rotates with a constant angular velocity ω. Determine the relationship between ω and the angle φ that is formed by the rod with length a and the vertical.

71. Consider a load of weight P at a point F on the platform of a balance (Fig. 30). $AB = a$, $BC = b$, $CD = c$, $IK = d$, and the length of the platform EG is l. Determine

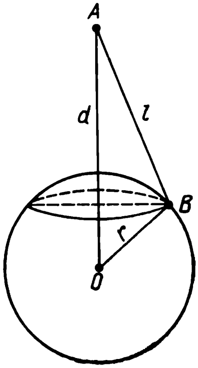

Fig. 28

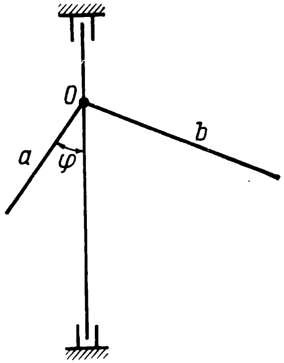

Fig. 29

the relationship between b, c, d, and l for the case when the weight P_{w} which compensates the weight of the load does not depend on the position of point F. Also determine the actual value of P_{w} in this case.

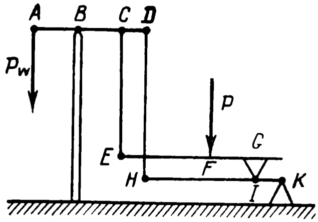

Fig. 30

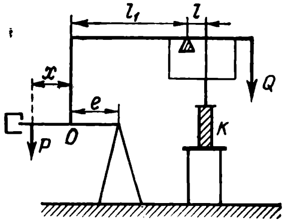

Fig. 31

72. A schematic drawing of a device for measuring elastic constants of solids is depicted in Fig. 31. Find the relationship between the force F that is applied to the sample K and the distance from the weight P to its zero point O, if weight Q balances the device in such a way that all the arms are in a horizontal position when the weight P is at its zero point and when there are no stresses in the sample.

73. Two rods AB and OC are connected at a right angle at a point C (Fig. 32). OC can rotate about a horizontal axis that passes through O; $AC = CB = a$ and $OC = b$. The points A and B are loaded by weights P_1 and P_2, respectively. Find the angle that AB forms with the horizontal plane in equilibrium if the weight of both rods is $2p$ per unit of length.

74. Three rods of equal length, $AB = BC = DC = a$, are connected at right angles at points B and C (Fig. 33).

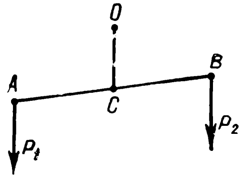

Fig. 32

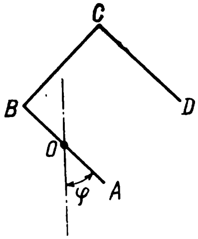

Fig. 33

The rod AB can rotate about a horizontal axis O that divides the rod in half. Determine the positions of stable and unstable equilibrium of the system if the weight of the unit length of the rod is p.

75. Find the Poisson bracket for the Cartesian components of the linear momentum $\mathbf{p}$ and angular momentum $\mathbf{L} = [\mathbf{r} \times \mathbf{p}]$.

76. Find the Poisson bracket for the components of the vector of angular momentum $\mathbf{L} = \sum_{i=1}^{n} [\mathbf{r}_i \times \mathbf{p}_i]$ for a system of n particles.

77. Show that $\{\varphi, L_z\} = 0$, where φ is an arbitrary function of the position and momentum of a particle.

78. Find the condition for the linear transformations of p and q

$$Q = aq + bp$$
$$P = cq + dp$$

to be canonical.

79. Show that $\{\mathbf{f}, L_z\} = [\mathbf{n} \times \mathbf{f}]$, where $\mathbf{f}$ is an arbitrary function of the position and momentum of a particle and $\mathbf{n}$ is a unit vector in the direction of the z-axis.

80. Construct the Lagrangian and the Hamiltonian for an electrically charged harmonic oscillator that is located in a homogeneous and stationary magnetic field $\mathbf{B}$ ($\mathbf{B} = \mu_0 \mathbf{H}$, where μ_0 is the permeability of empty space).

81. Construct the Lagrangian and the Hamiltonian for a system of two particles interacting via the Coulomb law. Express both functions in terms of the centre of mass coordinates and the separation between the particles.

82. Construct the Lagrangian and the Hamiltonian for a system of n particles interacting via the Coulomb law. The system is in an external electromagnetic field.

83. Find the Lagrangian and the Hamiltonian for a system of two oppositely charged particles placed in a homogeneous magnetic field. Show that by adding to the Lagrangian the total time derivative of a specially chosen function of coordinates, we can make the coordinates of the centre of mass cyclic (ignorable). What integral of motion corresponds to these cyclic coordinates?

84. Construct the Hamiltonian for a symmetrical top with one fixed point in the field of the earth's gravity.

85. Consider a thin disc of mass M sliding along a perfectly smooth horizontal plane. A particle of mass m moves across the disc. In a coordinate system that is attached to the disc and whose origin lies in the centre of the disc, the motion of the particle is governed by the law $x = x(t)$ and $y = y(t)$. Find the angular velocity of the disc as a function of time if at the initial instant of time the disc was motionless.

86. On the disc of Problem 85 the same particle of mass m moves with a velocity αt (relative to the disc) along a circumference of radius R. Find the motion equations.

87. Determine the inertia tensor, relative to the centre of mass, for the following molecules:

(1) CH_4; its structure is represented by the regular tetrahedron (a four-faced polyhedron with all of its faces equilateral triangles) with the carbon atom C at its centre and the four hydrogen atoms H at its vertices. The distance CH is $a = 1.07$ Å;

(2) H_2O; its structure is depicted in Fig. 34*a*;

(3) NH_3; its structure is depicted in Fig. 34*b*.

88. Prove that for a diatomic molecule the inertia tensor, calculated in the centre-of-mass reference frame, is determined only by one quantity, i.e. $I = \mu a^2$, where μ is the reduced mass of this diatomic system and a the distance between the positions of equilibrium of the atoms.

89. The moment of inertia of a molecule of hydrogen fluoride HF is 1.37×10^{-40} g cm^2 if calculated relative to the centre of mass. Determine the distance between the atoms of hydrogen and fluorine. The atomic mass of hydrogen is $M_H = 1.67 \times 10^{-24}$ g and that of fluorine is $M_F = 3.17 \times 10^{-22}$ g.

90. Determine the ratio between the moments of inertia of the molecules H_2, HD, and D_2 and also the ratio between their frequencies of vibrations, assuming that the interatomic potentials do not depend on the isotopic composition of the molecules.

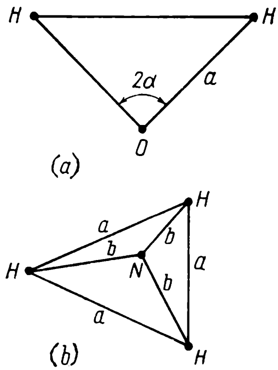

Fig. 34

91. For the following continuous rigid bodies, each of mass M, find the principal sets of axes, based at the corresponding centres of mass, and the principal moments of inertia:

(1) a rod in the form of a right parallelepiped with edges of length a, b, and c;

(2) a ball of radius R;

(3) a circular cone with an altitude h and the radius of the base R;

(4) an ellipsoid with the semiaxes a, b, and c;

(5) a spherical shell with an inner diameter d and an outer diameter D;

(6) a torus with a cross-section radius r and a middle radius R;

(7) a cylindrical pipe of length l with an inner radius r and an outer radius R;

(8) a right triangular prism with an altitude l and the side of the triangle a;

(9) a right hexagonal prism with an altitude l and the side of the hexagon a.

92. Consider a symmetrical top along whose axis of symmetry there acts a constant moment $\mathbf{N}$ of external forces. Construct the motion equation of such a top and solve it. The resultant of the external forces is zero.

93. Determine the components of angular velocity as functions of the angle of proper rotation. For a homogeneous

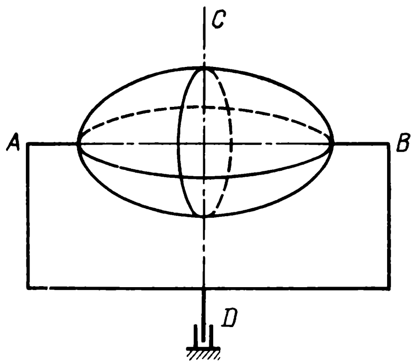

Fig. 35

ellipsoid that rotates about one of its axis, AB, which in turn rotates about the axis CD (Fig. 35) that is perpendicular to AB, find the maximal and minimal values of these components in a system of coordinates whose axes are taken along the principal axes. CD passes through the centre of the ellipsoid.

94. Solve Problem 93 under the conditions that AB forms an angle α with CD and the ellipsoid is symmetric about the AB-axis (Fig. 36).

95. Construct the Lagrangian for a homogeneous cylinder of radius a that rolls along the inner surface of a cylinder of radius R (Fig. 37).

96. A right and homogeneous cylinder of mass M, length l, and radius r rotates about a vertical axis OZ with a constant angular velocity ω. In the process the angle between the cylinder's axis of symmetry and OZ (OZ passes through the centre of mass of the cylinder) maintains a constant

value α; the distance between the thrust bearing and the bearing at the top (see Fig. 38) is h. Find the lateral pressure on both bearings.

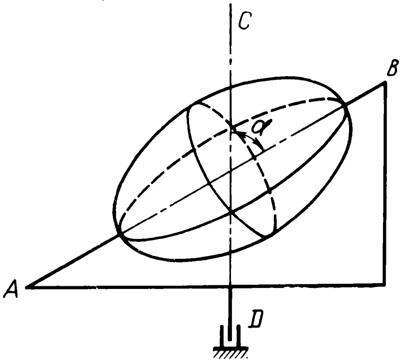

Fig. 36

97. Construct the motion equation in the form of Euler's equations (see the last three formulas of (I-32) on p. 20) for a symmetrical top in the earth's gravitational field.

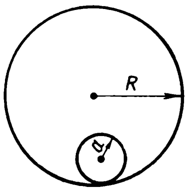

Fig. 37

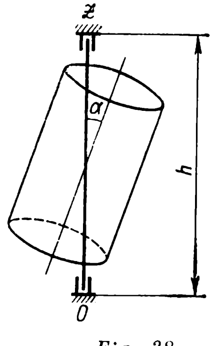

Fig. 38

98. Find the Euler angles as functions of time for the free rotation of a symmetrical top.

99. Find the elements of the rotation matrix that represents three successive rotations in the positive (counterclockwise) direction: first, through an angle θ about the

x-axis; then, through an angle ψ about the new y'-axis; last, through φ about the new z''-axis.

100. Express the components of angular velocity through the angles of Problem 99 and their time derivatives.

101. Figure 39 illustrates the principle of a monorail. C is the body of the coach of mass M_1, which moves uniformly in a straight line, and R is a frame that can freely rotate about a horizontal axis A_1A_2. A counterbalance of mass M_3 is rigidly attached to the frame. A flywheel of mass M_2 rotates freely about an axis B_1B_2 in bearings. The centre of mass of the flywheel lies at the intersection point O_1 of A_1A_2 and B_1B_2.

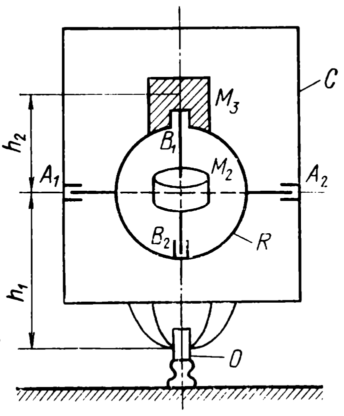

Fig. 39

Show that the solution $\varphi = \omega_0 t$, $\theta = \psi = 0$ is a stable solution of the motion equation of the monorail coach. Here φ is the rotation angle of the flywheel, ψ the deflection angle of the frame R (with the counterbalance) relative to the body C, θ the inclination angle between the body's axis and the vertical line. The flywheel's centre of mass is at a distance h_1 from point O, the body's centre of mass is at a distance l from the same point, and the centre of mass of the counterbalance is h_2 higher than the flywheel's centre of mass.

102. An elliptic, nonhomogeneous cylinder of length h, made out of two kinds of materials of densities ρ_1 and ρ_2, lies on a horizontal surface (Fig. 40). Determine the states of stable and unstable equilibrium. The lengths of the semimajor and semiminor axes are a and b, respectively.

103. Find the components of the strain tensor in spherical and cylindrical coordinates. (See Appendix 2.)

104. Find the deformation, i.e. the displacement vector $\mathbf{u}$, for a cylinder that is rotating about its axis with a constant angular velocity ω. (For the body force $\mathbf{f}$ in equation (I-50) take the centrifugal force per unit mass.)

105. Construct the dispersion equation for elastic waves in a single crystal with a cubic lattice. Find the phase velocities of such waves if the waves propagate parallel to the faces of the crystal, and if the waves propagate perpendicular to the faces.

106. A plane, longitudinal, and monochromatic wave falls at an angle of incidence θ_0 on the plane interface bet-

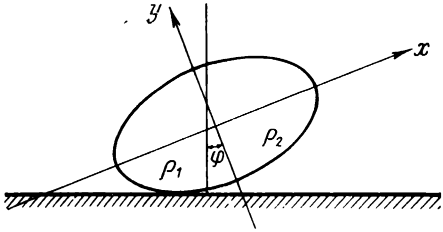

Fig. 40

ween a vacuum and a solid (Fig. 41). Find the laws of reflection and calculate the ratio between the perpendicular component of the energy flux of the reflected longitudinal wave

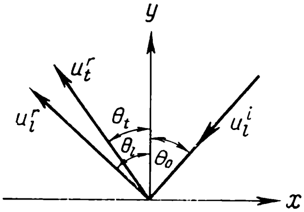

Fig. 41

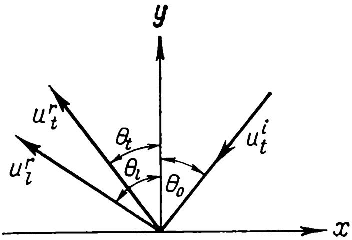

Fig. 42

and the same component of the energy flux of the incident wave. Calculate the ratio for the perpendicular component of the reflected transverse wave. In Fig. 41 u_l^i is the displacement of the longitudinal incident wave, u_l^r the displacement of the longitudinal reflected wave, and u_t^r the displacement of the reflected transverse wave.

107. Solve Problem 106 for the case when the incident wave is transverse and its plane of oscillations coincides with the incident plane (Fig. 42).

108. An incompressible viscous liquid is flowing under the influence of a pressure drop $\Delta p = p_2 - p_1$ between two infinite, parallel plates that are at a distance d from one another. Find the field of velocities and pressure distribution between the plates.

109. Proceeding from the conditions of Problem 108, find the heat flux and temperature distribution between the plates when (1) the temperature of the lower and upper plates is held at a constant value T_0 by means of a thermostat; (2) the lower plate is adiabatically isolated and the upper plate is held at a constant temperature.

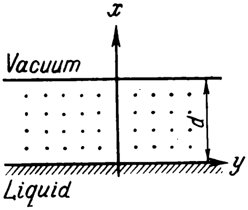

Fig. 43

110. Consider two coaxial cylinders with radiuses r_1 and r_2 ($r_1 > r_2$). Find the field of velocities in an incompressible viscous liquid between the cylinders for the following cases: (1) the outer cylinder rotates about its axis with an angular velocity ω_1; (2) the inner cylinder rotates with an angular velocity ω_2; (3) the outer and inner cylinders rotate in the same direction with angular velocities ω_1 and ω_2, respectively.

111. Under the conditions of Problem 110 find the temperature distribution in the liquid if the rotating inner cylinder is thermally isolated and the motionless outer cylinder is held at a constant temperature T_0.

112. Write the right part of the Navier-Stokes equation in cylindrical and spherical coordinates.

113. An infinite plate of thickness d is the interface between a vacuum and a liquid (Fig. 43). Find the natural frequencies ω for longitudinal sound vibrations assuming that all quantities depend on the transverse coordinate x only.

114. Find the dispersion equation for the propagation of an elastic wave $\mathbf{u} = \mathbf{f}(z)\, e^{i(\omega t - kx)}$ in an isotropic solid (the Rayleigh wave). Here $\mathbf{f}(z)$ is a damped function. Study the structure of this wave.

115. Determine the natural frequencies of the radial vibrations of an elastic sphere with a radius R placed in a vacuum.

116. Determine the frequency of the radial vibrations of a spherical cavity in an infinite elastic medium.

117. Determine the natural frequencies of the sound vibrations of a gas that fills a closed section of a rectangular pipe of dimensions a, b, and d. Assume the pipe is rigid.

118. Construct the wave equation and find the dispersion equation for sound waves generated by a source moving with a constant velocity $\mathbf{v}_0$ relative to a liquid (gas). Find the frequency registered by a receiver that is motionless with respect to the liquid (gas).

119. What frequency will be registered by a receiver that moves in a liquid (gas) relative to a stationary source of sound (Doppler effect)?

120. Find the field of velocities in a liquid that is flowing through a ring-shaped pipe (the inner and outer radiuses are r_1 and r_2, respectively).

121. An incompressible viscous liquid is flowing under the influence of a pressure drop $\Delta p = p_2 - p_1$ in a pipe with an elliptic cross section. The length of the pipe is l. Determine the field of velocities and the amount of liquid that flows through the cross section in a unit of time.

122. An incompressible viscous liquid is flowing under the influence of a pressure drop $\Delta p = p_2 - p_1$ in a pipe with a circular cross section. Find the temperature distribution if the temperature of the pipe is held at a constant value T_0.

SECTION II

Electrodynamics

The two fundamental quantities that characterize an electromagnetic field are the electric field vector **E** and the vector of magnetic induction **B** (the magnetic flux density). The electric field vector is determined from the force acting on a test charge e, i.e.

$$\mathbf{F} = e\mathbf{E} \tag{II-1}$$

and the magnetic induction is determined from the force acting on a circuit element $d\mathbf{l}$ which carries a current I, i.e.

$$d\mathbf{F} = I\,[d\mathbf{l} \times \mathbf{B}] \tag{II-2}$$

When considering an electromagnetic field in a dielectric or magnetic material, where the field is created by external sources or by the polarization and magnetization of the material, it is convenient to introduce two supplementary characteristics of the electromagnetic field: the electric displacement vector

$$\mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P} \tag{II-3}$$

and the magnetic field strength (or intensity)

$$\mathbf{H} = \frac{\mathbf{B}}{\mu_0} - \mathbf{M} \tag{II-4}$$

where **P** is the polarization vector representing the electric dipole moment per unit volume, **M** is the magnetization vector representing the magnetic dipole moment per unit volume, and ε_0 and μ_0 are the permittivity and permeability of empty space, respectively:

$$\varepsilon_0 = 8.854 \times 10^{-12}\ \mathrm{F\,m^{-1}} \tag{II-5}$$

$$\mu_0 = 4\pi \times 10^{-7}\ \mathrm{H\,m^{-1}} \tag{II-6}$$

$$\varepsilon_0\mu_0 = c^{-2}$$

where c is the velocity of light in empty space.

The relationships between **P** and **E**, **M** and **H** are determined by the properties of the medium. For an anisotropic medium in the approximation linear with respect to the field and for fields slowly varying in space and time the following relationships hold true:

$$P_i = \varepsilon_0 \alpha_{ik} E_k \tag{II-7}$$

$$M_i = \chi_{ik} H_k \tag{II-8}$$

The indexes i and k assume the values 1, 2, and 3 and denote the projections of a vector (**P** or **M**) on the x, y, and z axes. In all formulas where the same index appears twice we shall automatically sum over that index. In (II-7) α_{ik} is the polarizability tensor of the medium and in (II-8) χ_{ik} is the magnetic susceptibility tensor of the medium.

Through the use of (II-3), (II-4), (II-7), and (II-8) we can obtain the relationships between **D** and **E**, **B** and **H**:

$$D_i = \varepsilon_0 \varepsilon_{ik} E_k \tag{II-9}$$

$$B_i = \mu_0 \mu_{ik} H_k \tag{II-10}$$

where

$$\varepsilon_{ik} = \delta_{ik} + \alpha_{ik}$$

is the permittivity tensor, and

$$\mu_{ik} = \delta_{ik} + \chi_{ik}$$

is the permeability tensor.

In an isotropic medium the relationships (II-7)-(II-10) are simplified thus: C

$$\mathbf{P} = \varepsilon_0 \alpha \mathbf{E} \tag{II-11}$$

$$\mathbf{M} = \chi \mathbf{H} \tag{II-12}$$

$$\mathbf{D} = \varepsilon \varepsilon_0 \mathbf{E} \tag{II-13}$$

$$\mathbf{B} = \mu \mu_0 \mathbf{H} \tag{II-14}$$

where $\varepsilon = 1 + \alpha$ is simply the permittivity of the medium and $\mu = 1 + \chi$ the permeability.

A set of four equations called the Maxwell equations are the starting equations of electrodynamics. In the

integral form they are:

$$\oint H_l\, dl = \int j_n\, dS + \int \left(\frac{\partial \mathbf{D}}{\partial t}\right)_n dS \qquad \text{(II-15)}$$

$$\oint E_l\, dl = -\int \left(\frac{\partial \mathbf{B}}{\partial t}\right)_n dS \qquad \text{(II-16)}$$

$$\oint D_n\, dS = \int \rho\, dV \qquad \text{(II-17)}$$

$$\oint B_n\, dS = 0 \qquad \text{(II-18)}$$

where $\mathbf{j}$ is the conduction current density and ρ the electric charge density.

In the differential form the Maxwell equations are:

$$\operatorname{curl} \mathbf{H} = \mathbf{j} + \frac{\partial \mathbf{D}}{\partial t} \qquad \text{(II-19)}$$

$$\operatorname{curl} \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \text{(II-20)}$$

$$\operatorname{div} \mathbf{D} = \rho \qquad \text{(II-21)}$$

$$\operatorname{div} \mathbf{B} = 0 \qquad \text{(II-22)}$$

At a boundary between two media the electromagnetic field vectors are subject to the following boundary conditions:

$$D_{2n} - D_{1n} = \sigma \qquad \text{(II-23)}$$

$$E_{1t} = E_{2t} \qquad \text{(II-24)}$$

$$B_{1n} = B_{2n} \qquad \text{(II-25)}$$

$$[\mathbf{n} \times (\mathbf{H}_2 - \mathbf{H}_1)] = \mathbf{i} \qquad \text{(II-26)}$$

where $\mathbf{n}$ is a vector normal to the boundary and directed from medium *1* to medium *2*, σ the surface charge density, $\mathbf{i}$ the surface current per unit length flowing along the boundary, E_t the tangential component of the electric field at the boundary, and D_n and B_n are the normal components of the electric displacement and the magnetic induction, respectively, at the boundary.

Electrostatics. In this branch of electrodynamics all quantities are constant in time and electric charges are stationary. From the Maxwell equations (II-19)-(II-22)

we then have the system of equations of electrostatics:

$$\operatorname{curl} \mathbf{E} = 0 \tag{II-27}$$

$$\operatorname{div} \mathbf{D} = \rho \tag{II-28}$$

Equation (II-27) is satisfied if we introduce the electrostatic potential by the relation

$$\mathbf{E} = -\operatorname{grad} \varphi \tag{II-29}$$

In a homogeneous medium the potential satisfies the Poisson equation

$$\Delta\varphi = -\frac{\rho}{\varepsilon\varepsilon_0} \tag{II-30}$$

or when there is no charge the Laplace equation

$$\Delta\varphi = 0$$

At an interface of two media with permittivities ε_1 and ε_2 the potential is subject to the following boundary conditions:

$$\varphi_1 = \varphi_2, \tag{II-31}$$

$$\varepsilon_1 \left(\frac{\partial\varphi}{\partial n}\right)_1 - \varepsilon_2 \left(\frac{\partial\varphi}{\partial n}\right)_2 = \frac{\sigma}{\varepsilon_0} \tag{II-32}$$

In electrostatics the electric field does not penetrate a conductor, and therefore from equation (II-29) it follows that inside a conductor the potential is constant.

Since $\frac{\partial\varphi}{\partial n} = 0$ inside a conductor, from the boundary condition (II-32) we can find the relationship between the surface charge density (on the surface of the conductor) and the potential near the conductor:

$$\sigma = -\varepsilon_0\varepsilon \left(\frac{\partial\varphi}{\partial n}\right)_S \tag{II-33}$$

The total charge of the conductor is

$$e = -\oint \varepsilon_0\varepsilon \frac{\partial\varphi}{\partial n}\, dS \tag{II-34}$$

where the integral is taken over the conductor's surface S.

For a system of more than one conductor there are usually two types of problems:

(1) given the potential on every conductor, i.e.

$$\varphi(\mathbf{r})|_{S_i} = \varphi_i \tag{II-35}$$

find the potential at any point in the system;

(2) given the charge of every conductor q_i i.e.

$$\varphi(\mathbf{r})|_{S_i} = \text{constant}$$

$$-\oint\limits_{S_i} \varepsilon \frac{\partial \varphi}{\partial n} dS_i = \frac{q_i}{\varepsilon_0} \tag{II-36}$$

find the potential at any point in the system.

Many problems in electrostatics that possess cylindrical symmetry can be solved through the use of functions of a complex variable. The real and imaginary parts of an analytic function $W(z) = \varphi + i\Psi$ of a complex variable $z = x + iy$ are solutions of the Laplace equation: $\Delta\varphi = 0$ and $\Delta\Psi = 0$. Besides, φ and Ψ are restricted by the condition

$$\frac{\partial \varphi}{\partial x}\frac{\partial \Psi}{\partial x} + \frac{\partial \varphi}{\partial y}\frac{\partial \Psi}{\partial y} = 0$$

i.e. the curves φ = constant and Ψ = constant are mutually orthogonal. Thus φ and Ψ can be the solutions of the electrostatic problem. If $\varphi(x, y)$ is the potential, the curves Ψ = constant are the lines of force orthogonal to the equipotential surfaces. Let us put $\varphi = 0$ on a smooth contour L. In order to find the equipotential surfaces φ = constant, we must write the equation for the contour L in the parametric form

$$x = f(P), \quad y = F(P) \tag{II-37}$$

where f and F are single-valued functions and where the range of admissible values of P corresponds to the movement of the point (x, y) along the grounded conductor ($\varphi = 0$). We must seek the potential in the form

$$\varphi = \operatorname{Im} W \tag{II-38}$$

and W is found from the equation $z = f(W) + iF(W)$.

If the distribution of body and surface charges is given, the potential at point $\mathbf{r}$ is

$$\varphi(\mathbf{r}) = \frac{1}{4\pi\varepsilon\varepsilon_0}\int \frac{\rho(\mathbf{r}')\,dV'}{|\mathbf{r}-\mathbf{r}'|} + \frac{1}{4\pi\varepsilon\varepsilon_0}\int \frac{\sigma(\mathbf{r}')\,dS'}{|\mathbf{r}-\mathbf{r}'|} \tag{II-39}$$

At great distances, i.e. distances that are much bigger than the dimensions of the system, the potential can be represented as a series

$$\varphi(\mathbf{r}) = \varphi^{(0)}(\mathbf{r}) + \varphi^{(1)}(\mathbf{r}) + \varphi^{(2)}(\mathbf{r}) + \dots \qquad \text{(II-40)}$$

where $\varphi^{(0)}(\mathbf{r})$ is the potential produced by a point charge equal to the total charge of the system:

$$\varphi^{(0)}(\mathbf{r}) = \frac{e}{4\pi\varepsilon\varepsilon_0 r} \qquad \text{(II-41)}$$

$\varphi^{(1)}(\mathbf{r})$ the potential produced by a dipole with the dipole moment $\mathbf{p} = \int \rho(\mathbf{r})\,\mathbf{r}\,dV$:

$$\varphi^{(1)}(\mathbf{r}) = \frac{(\mathbf{p}\cdot\mathbf{r})}{4\pi\varepsilon\varepsilon_0 r^3} \qquad \text{(II-42)}$$

and $\varphi^{(2)}(\mathbf{r})$ the potential produced by a quadrupole:

$$\varphi^{(2)}(\mathbf{r}) = \frac{1}{4\pi\varepsilon\varepsilon_0} \sum_{i,\,k} \frac{1}{2!} Q_{ik} \frac{\partial^2}{\partial x_i\,\partial x_k}\left(\frac{1}{r}\right) \qquad \text{(II-43)}$$

The components of the tensor of quadrupole moment are defined as follows:

$$Q_{ik} = \int \rho(\mathbf{r})\,(3x_i x_k - r^2\delta_{ik})\,dV \qquad \text{(II-44)}$$

The energy of an electrostatic field is

$$W = \frac{1}{2}\int \rho\varphi\,dV = \frac{1}{2}\int (\mathbf{D}\cdot\mathbf{E})\,dV \qquad \text{(II-45)}$$

The interaction energy of two electrically charged systems with charge densities ρ_1 and ρ_2 is determined thus:

$$W = \int \rho_1\varphi_2\,dV = \frac{1}{4\pi\varepsilon\varepsilon_0}\int \frac{\rho_1(\mathbf{r}_1)\,\rho_2(\mathbf{r}_2)\,dV_1\,dV_2}{|\mathbf{r}_1 - \mathbf{r}_2|} \qquad \text{(II-46)}$$

Magnetostatics. Phenomena that take place in a constant magnetic field, i.e. when the magnetic field strength and the magnetic induction are independent of time, are governed by the system of equations of magnetostatics:

$$\operatorname{curl}\mathbf{H} = \mathbf{j} \qquad \text{(II-47)}$$

$$\operatorname{div}\mathbf{B} = 0 \qquad \text{(II-48)}$$

together with the boundary conditions (II-25) and (II-26). Equation (II-48) is satisfied if we introduce the vector potential by the relation

$$\mathbf{B} = \operatorname{curl} \mathbf{A} \tag{II-49}$$

In a homogeneous and isotropic medium the vector potential **A** satisfies the equation

$$\Delta \mathbf{A} = -\mu\mu_0 \mathbf{j} \tag{II-50}$$

The solution of this equation for a given current density **j** has the form

$$\mathbf{A}(\mathbf{r}) = \frac{\mu\mu_0}{4\pi} \int \frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} dV' \tag{II-51}$$

At distances considerably greater than the dimensions of the system

$$\mathbf{A}(\mathbf{r}) = \frac{\mu\mu_0 [\mathbf{m}\times\mathbf{r}]}{4\pi r^3} \tag{II-52}$$

where **m** is the magnetic moment of the system equal to

$$\mathbf{m} = \frac{1}{2} \int [\mathbf{r}\times\mathbf{j}(\mathbf{r})]\, dV \tag{II-53}$$

The energy of a stationary magnetic field is

$$W = \frac{1}{2}\int (\mathbf{B}\cdot\mathbf{H})\, dV = \frac{1}{2}\int (\mathbf{A}\cdot\mathbf{j})\, dV$$
$$= \frac{\mu\mu_0}{8\pi} \int \frac{\mathbf{j}(\mathbf{r})\,\mathbf{j}(\mathbf{r}')\, dV\, dV'}{|\mathbf{r}-\mathbf{r}'|} \tag{II-54}$$

For a system of conductors the energy of the magnetic field can be rewritten as

$$W = \frac{1}{2} \sum_{i,\,k} L_{ik} I_i I_k \tag{II-55}$$

where

$$L_{ik} = \frac{\mu\mu_0}{4\pi I_k I_i} \int \frac{\mathbf{j}_k(\mathbf{r}_k)\,\mathbf{j}_i(\mathbf{r}_i)\, dV_k dV_i}{|\mathbf{r}_k - \mathbf{r}_i|} \tag{II-56}$$

The factors L_{ik} for $i \neq k$ are called the mutual inductances, and for $i = k$ the self-inductances.

Quasi-stationary fields. In the case of fields slowly varying in time (i.e.

$$\omega \ll \frac{\sigma^*}{\varepsilon} \text{ and } l \ll \lambda \tag{II-57}$$

where σ^* is the electrical conductivity of the medium; ω and λ are the frequency and the wavelength of the electromagnetic oscillations, respectively; and l is the characteristic length of the system) the Maxwell equations take the form

$$\left.\begin{aligned} \operatorname{curl} \mathbf{H} &= \mathbf{j} \\ \operatorname{curl} \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\ \operatorname{div} \mathbf{D} &= \rho \\ \operatorname{div} \mathbf{B} &= 0 \end{aligned}\right\} \qquad \text{(II-58)}$$

Electromagnetic waves. In the general case of varying fields we must solve the system of equations (II-19)-(II-22). When there are no charges and conduction currents, this system describes an electromagnetic field in free space. Plane monochromatic waves, in which

$$\mathbf{E} = \mathbf{E}_0 e^{i\mathbf{kr} - i\omega t} \qquad \text{(II-59)}$$

$$\mathbf{H} = \mathbf{H}_0 e^{i\mathbf{kr} - i\omega t} \qquad \text{(II-60)}$$

are a partial solution for the Maxwell equations for a free field. Here ω is the frequency of electromagnetic oscillations and $\mathbf{k}$ is the wave vector. In an isotropic medium the direction of $\mathbf{k}$ coincides with the direction of the propagation of the wave's energy. In magnitude $k = \omega/v$, where $v = c(\varepsilon\mu)^{-1/2}$ is the phase velocity of the wave.

The density of the energy flux of an electromagnetic field is defined by the Poynting vector

$$\mathbf{S} = [\mathbf{E} \times \mathbf{H}] \qquad \text{(II-61)}$$

For varying electromagnetic fields we can express the relationship between the electric field intensity and the magnetic induction, on the one hand, and the potentials, on the other, in this way:

$$\mathbf{E} = -\operatorname{grad} \varphi - \frac{\partial \mathbf{A}}{\partial t} \qquad \text{(II-62)}$$

$$\mathbf{B} = \operatorname{curl} \mathbf{A} \qquad \text{(II-63)}$$

If we impose the Lorentz condition

$$\operatorname{div} \mathbf{A} + \frac{\varepsilon\mu}{c^2} \frac{\partial \varphi}{\partial t} = 0 \qquad \text{(II-64)}$$

the potentials φ and $\mathbf{A}$ satisfy the equations

$$\Delta\varphi - \frac{\varepsilon\mu}{c^2}\frac{\partial^2\varphi}{\partial t^2} = -\frac{\rho}{\varepsilon\varepsilon_0} \tag{II-65}$$

$$\Delta\mathbf{A} - \frac{\varepsilon\mu}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = -\mu\mu_0\mathbf{j} \tag{II-66}$$

We can write the solutions of (II-65) and (II-66) in the form of retarded potentials

$$\varphi(\mathbf{r}, t) = \frac{1}{4\pi\varepsilon\varepsilon_0}\int \frac{\rho\left(\mathbf{r}', t - \frac{|\mathbf{r}-\mathbf{r}'|}{v}\right)}{|\mathbf{r}-\mathbf{r}'|}\, dV' \tag{II-67}$$

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu\mu_0}{4\pi}\int \frac{\mathbf{j}\left(\mathbf{r}', t - \frac{|\mathbf{r}-\mathbf{r}'|}{v}\right)}{|\mathbf{r}-\mathbf{r}'|}\, dV' \tag{II-68}$$

In a vacuum and at distances from a system of charges which are considerably greater than the length of the electromagnetic wave radiated by this system, $r \gg \lambda$, we get

$$\mathbf{B} = \frac{1}{c}[\dot{\mathbf{A}} \times \mathbf{n}] \tag{II-69}$$

$$\mathbf{E} = c\,[\mathbf{B} \times \mathbf{n}] = [[\dot{\mathbf{A}} \times \mathbf{n}] \times \mathbf{n}] \tag{II-70}$$

where

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi r}\int \mathbf{j}\left(\mathbf{r}', t - \frac{r - (\mathbf{r}'\cdot\mathbf{n})}{c}\right) dV' \tag{II-71}$$

and $\mathbf{n} = \frac{\mathbf{r}}{r}$ is a unit vector in the direction of wave propagation.

If, in addition, the wavelength is much greater than the dimensions of the radiating system of charges, the electromagnetic field at great distances can be represented as a sum of fields generated by a dipole, quadrupole and other multipoles. The dipole radiation has the maximum intensity. The corresponding field is determined by the relationships

$$\left.\begin{aligned} \mathbf{B} &= \frac{\mu_0[\ddot{\mathbf{p}} \times \mathbf{n}]}{4\pi c r} \\ \mathbf{E} &= \frac{\mu_0}{4\pi r}[[\ddot{\mathbf{p}} \times \mathbf{n}] \times \mathbf{n}] \end{aligned}\right\} \tag{II-72}$$

where $\mathbf{p}$ is the dipole moment.

The intensity of the dipole radiation is determined by the formula

$$J=\frac{(\ddot{p})^2}{6\pi\varepsilon_0 c^3} \tag{II-73}$$

Magnetohydrodynamics. This branch of electrodynamics studies the effect of electromagnetic fields on electrically conducting fluids. In the hydrodynamic approximation the motion of a system is described by the variations in the density, velocity, and pressure. At low frequencies the electromagnetic field satisfies the equations (II-58). The equations of magnetohydrodynamics look like this:

$$\frac{\partial\rho_m}{\partial t}+\operatorname{div}\rho_m\mathbf{v}=0 \tag{II-74}$$

$$\rho_m\frac{\partial\mathbf{v}}{\partial t}+\rho_m(\mathbf{v}\cdot\nabla)\mathbf{v}=-\operatorname{grad}p+[\mathbf{j}\times\mathbf{B}]+\eta\Delta\mathbf{v}+\rho_m\mathbf{g} \tag{II-75}$$

$$\operatorname{curl}\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t} \tag{II-76}$$

$$\operatorname{curl}\mathbf{H}=\mathbf{j} \tag{II-77}$$

$$\mathbf{j}=\sigma^*(\mathbf{E}+[\mathbf{v}\times\mathbf{B}]) \tag{II-78}$$

where ρ_m is the density of the medium, p the pressure, η the viscosity, $\mathbf{g}$ the acceleration due to gravity, σ^* the electrical conductivity of the medium, and v the velocity.

Equations (II-74)-(II-78) must be supplemented by the state equations of the medium.

Let us consider in brief the **special theory of relativity.** The fundamental principles of the special theory of relativity are:

(1) the velocity of light in free space is the same in any inertial frame of reference and equal to $c = 2.99793 \times 10^8$ m s^{-1};

(2) the laws of physics act in all inertial reference frames in the same way.

These principles have the following corollaries.

Consider an event with coordinates $x_1 = x$, $x_2 = y$, $x_3 = z$, $x_4 = ict$ in a reference frame K and with coordinates $x_1' = x'$, $x_2' = y'$, $x_3' = z'$, $x_4' = ict'$ in another reference frame K'. If K' moves along the x-axis of K with a velocity v in relation to K, the coordinates of the event

in the two reference frames are linked by the Lorentz transformation

$$\left.\begin{aligned} x_1' &= \frac{x_1 + i\beta x_4}{\sqrt{1-\beta^2}} \\ x_2' &= x_2 \\ x_3' &= x_3 \\ x_4' &= \frac{x_4 - i\beta x_1}{\sqrt{1-\beta^2}} \end{aligned}\right\} \qquad \text{(II-79)}$$

where $\beta = v/c$.

Assume that a body moves with a velocity $\mathbf{u}$ in relation to K. The velocity in relation to K' will then be

$$u_x' = \frac{u_x - v}{1 - \frac{vu_x}{c^2}}, \quad u_y' = \frac{u_y\sqrt{1-\beta^2}}{1 - \frac{vu_x}{c^2}}, \quad u_z' = \frac{u_z\sqrt{1-\beta^2}}{1 - \frac{vu_x}{c^2}} \qquad \text{(II-80)}$$

When a force $\mathbf{F}$ acts on a particle, the law of motion in the differential form is

$$\frac{d\mathbf{p}}{dt} = \mathbf{F} \qquad \text{(II-81)}$$

where

$$\mathbf{p} = \frac{m_0\mathbf{v}}{\sqrt{1-\beta^2}} \qquad \text{(II-82)}$$

is the momentum of the relativistic particle, and m_0 is its rest mass.

A totality of four numbers that transform according to a Lorentz transformation if we change the frame of reference is called a 4-vector. This may be the 4-momentum with the components $(\mathbf{p}, iE/c)$, where $\mathbf{p}$ is the conventional momentum and E is the energy; or a wave vector and frequency in a plane electromagnetic wave propagating in a vacuum $(\mathbf{k}, i\omega/c)$; or the 4-vector of current density $(\mathbf{j}, ic\rho)$; or the 4-vector potential $(\mathbf{A}, i\varphi/c)$.

A totality of sixteen numbers that transform according to a double Lorentz transformation if we change the frame of reference is called a 4-tensor of rank 2. This may be the electromagnetic field tensor, which can be represent-

ed as

$$F_{\alpha\beta}=\begin{pmatrix} 0 & cB_z & -cB_y & -iE_x \\ -cB_z & 0 & cB_x & -iE_y \\ cB_y & -cB_x & 0 & -iE_z \\ iE_x & iE_y & iE_z & 0 \end{pmatrix} \tag{II-83}$$

or the energy-momentum tensor

$$T_{\alpha\beta}=\varepsilon_0\left(F_{\alpha\mu}F_{\beta\mu}-\frac{1}{4}\delta_{\alpha\beta}F_{\mu\nu}^2\right) \tag{II-84}$$

$$(\alpha,\ \beta,\ \mu,\ \nu=1,\ 2,\ 3,\ 4)$$

PROBLEMS

Vector analysis

1. Calculate the gradient of a function, $f(r)$, that depends only on the absolute value of the radius vector $\mathbf{r}$.

2. Calculate div $\mathbf{r}$, curl $\mathbf{r}$, curl $\varphi(r)\,\mathbf{r}$.

3. Calculate grad $(\mathbf{P}\cdot\mathbf{r})$, grad $\frac{(\mathbf{P}\cdot\mathbf{r})}{r^3}$, $(\mathbf{P}\cdot\nabla)\,\mathbf{r}$, div $[\mathbf{P}\times\mathbf{r}]$, curl $[\mathbf{r}\times\mathbf{P}]$, where $\mathbf{P}$ is a constant vector.

4. Calculate grad $\mathbf{A}(r)\,\mathbf{B}(r)$, div $\varphi(r)\,\mathbf{A}(r)$, curl $(\varphi(r)\times\times\mathbf{A}(r))$. The functions $\varphi(r)$, $\mathbf{A}(r)$, and $\mathbf{B}(r)$ depend only on the absolute value of the radius vector $\mathbf{r}$.

5. Using Ostrogradski's theorem, calculate the integrals

$$\mathbf{I}=\oint\mathbf{r}(\mathbf{A}\cdot\mathbf{n})\,dS,\quad \mathbf{I}=\oint(\mathbf{A}\cdot\mathbf{r})\,\mathbf{n}\,dS$$

if the volume enclosed by the surface is V and if $\mathbf{A}$ is a constant vector.

6. Show that $\int\limits_V \mathbf{A}\,dV=0$ if inside the volume div $\mathbf{A}=0$ and on its boundary $A_n=0$.

7. Show that the divergence of vector

$$\mathbf{A}+\frac{1}{4\pi}\,\mathrm{grad}\int\frac{\mathrm{div}\,\mathbf{A}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,dV'$$

is zero.

8. Find a solution for the Laplace equation that depends only on the absolute value r of the radius vector.

9. Write the Maxwell equations (II-19) to (II-22) in (a) cylindrical coordinates; (b) spherical coordinates.

Electrostatics

10. Find the electric field intensity vector inside and outside a uniformly charged sphere of radius R. The body charge density of the sphere is ρ.

11. A sphere of radius R is uniformly charged with a charge density ρ. Inside it there is a spherical cavity of radius R' whose centre is at a distance a from the centre of the sphere. Find the electric field intensity vector inside the cavity and inside and outside the sphere.

12. Find the electric field vector inside and outside a sphere with a body charge density varying as follows:

$$\rho = ar^n$$

where $n > -2$. The radius of the sphere is R.

13. Find the electric field vector inside and outside a uniformly charged solid cylinder of radius R. The electric charge per unit length of the cylinder is $\varkappa$.

14. A layer of nonconducting matter is put between two parallel planes and is charged to a density ρ. Find the electric field vector inside and outside the layer if its thickness is d.

15. Find the capacitances of the following capacitors: (a) spherical, (b) plane-parallel, and (c) cylindrical. Between the plates of each capacitor there is a dielectric of permittivity ε.

16. Two long, cylindrical conductors are arranged parallel to each other at a distance d. Calculate the capacitance per unit length of the system provided $d \gg R_1$ and $d \gg R_2$, where R_1 and R_2 are the radiuses of the cylinders.

17. Find the equation of the lines of force for a system of two point charges e and $-e$ with a distance d between them.

18*. For the case of a homogeneous electric field with a field vector $\mathbf{E}$ write the corresponding complex-valued potential W. Consider the special case of the electric field of a charged plane with a surface charge density σ.

19. Determine the potential near a grounded angle formed by two planes $x = 0$ and $y = 0$.

* In Problems 18 to 24 we assume the distribution of the potentials to be two-dimensional.

20. Determine the equipotential surfaces and lines of force, if the potential is $\varphi = \operatorname{Re}(\sqrt{z})$. What grounded contour has such a potential?

21. Determine the potential and the equipotential surfaces if the complex-valued potential is $W = \ln z$.

22. Find the potential near a grounded parabola $y^2 = 4a(x + a)$.

23. Find the potential near a grounded ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Consider the limiting case of the potential near a circle (in the three-dimensional case, a cylinder), assuming $b = a$.

24. Find the equipotential surfaces and lines of force near a grounded hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

25. Find the potential and the electric field intensity vector on the axis of a flat ring that has a surface charge density σ (the inner radius of the ring is R_1 and the outer R_2). Consider the following limiting cases: (a) the field of a flat disc ($R_1 \to 0$) and (b) the field of a plane ($R_1 \to 0$, $R_2 \to \infty$).

26. Determine the potential of the electric field produced by the electron of a hydrogen atom, assuming that the electron charge in the ground state is distributed with a charge density $\rho = \frac{e}{\pi a^3} e^{-2r/a}$, where a is a constant.

27. Determine the Fourier transform for the potential of a point charge.

28. Find the potential of the electric field produced by a charge that is distributed in an infinite medium by the law $\rho = \rho_0 \sin ax \sin by \sin cz$.

29. A point charge e is situated at a distance d from a conducting plane that is grounded. Find the potential and the electric field intensity vector of the system. Determine the surface density of the charge that is induced on the grounded surface. Show that the total induced charge is equal to $-e$.

30. Using the method of images, find the potential of a charge q that is placed inside a right angle formed by two conducting planes.

31. A point charge e is at a distance d from the centre of a conducting sphere of radius R. Using the method of images, determine the electric potential of the system. The sphere is grounded.

32. Find the potential of a system consisting of a point charge e and an insulated conducting sphere of radius R near the charge.

33. A point charge e is at a distance d from the centre of a spherical projection of a conducting plane. The centre of the projection lies on the plane, and the charge is situated opposite the point on the projection that is farthest from the plane. Determine the potential of the system if the radius of the projection is R.

34. At a distance d from the centre of a conducting grounded sphere of radius R an electric dipole p is placed whose positive charge is closest to the sphere. Find the electric potential of this system.

35. An electric charge e is placed at a distance d from the flat surface of an infinite dielectric with a permittivity ε_2. The permittivity of the medium where the charge is located is ε_1. Determine the potential φ and the electric displacement vector $\mathbf{D}$ in the two media.

36. The centre of a conducting sphere of radius R is on the flat boundary between two dielectrics with permittivities ε_1 and ε_2. The charge of the sphere is e. Find the potential of this system, the electric displacement vector, and the charge distribution on the surface of the sphere.

37. Solve Problem 31 by an expansion of the potential in a series of solutions of the Laplace equation in spherical coordinates. Determine the surface charge density and the total charge induced on the sphere.

38. A point charge e is placed at a distance d from a conducting sphere of radius R that has a potential V. Find the potential outside the sphere and the surface charge density on the sphere.

39. Determine the potential of a charged sphere of radius R. The surface charge density varies according to the law $\sigma = \sigma_0 \cos \theta$.

40. Determine the potential and the electric field vector of a uniformly polarized ball of radius R. The polarization vector of the ball is $\mathbf{P}$.

41. A conducting sphere of radius R is located in a non-uniform electrostatic field. Determine the potential around the sphere.

42. A conducting sphere that is grounded is located in a uniform electric field $\mathbf{E}_0$. Find the potential of this system and the surface charge density of the sphere.

43. A sphere of radius R made of a dielectric material is located in a uniform electric field $\mathbf{E}_0$. Determine the potential inside and outside the sphere.

44. Consider a sphere that is electrically charged with a surface density σ everywhere except for a spherical segment near the pole. Determine the potential inside and outside the sphere if the segment is limited by a circle with $\theta = \alpha$.

45. One face of a rectangular parallelepiped has a potential V. All the rest have a zero potential. Find the potential inside the parallelepiped.

46. Two opposite faces, $z = 0$ and $z = c$, of a rectangular parallelepiped with edges a, b, and c, have potentials V_1 and V_2, respectively. The rest are grounded. Find the potential inside the parallelepiped.

47. A disc of radius R that has a surface charge density σ is placed coaxially in a hollow cylinder of radius r_0 with conducting walls. Find the potential inside the cylinder. Consider the limiting case of a point charge inside the cylinder, i.e. $R \to 0$ but the charge of the disc remains finite, or $\pi R^2 \sigma = e$.

48. Find the potential of a field that is produced by a point charge e and a homogeneous plane-parallel lamina that is at a distance d from the charge. The thickness of the lamina is a and its permittivity is ε. Consider the particular case of a point charge on the surface of a semi-infinite crystal, and compare the obtained solution with that of Problem 47.

49. Find the quadrupole moment of an ellipsoid that is uniformly charged and has a body charge density ρ.

50. Determine the potential of an electric field produced by a point charge e that is placed in a homogeneous and anisotropic medium with a given permittivity tensor.

51. Find the electric field intensity vector inside an anisotropic lamina made of a dielectric and placed in a uniform field $\mathbf{E}_0$.

52. Calculate the energy of the interaction between the electron cloud of a hydrogen atom and the proton (which is the nucleus of the hydrogen atom). The charge density inside the cloud is $\rho = \frac{e}{\pi a^3} e^{-2r/a}$, where a is the Bohr radius.

53. Calculate the energy of the interaction between two balls whose distribution of charges e_1 and e_2 is spherically symmetric. The distance between the centres of the balls is a.

54. Consider a sphere that sinks into a liquid to a depth less than half its diameter when it is not charged. What should be the charge of the sphere so that it sinks to a depth exactly half its diameter? The mass of the sphere is M, its radius is R, and the liquid has a density μ and a permittivity ε.

Direct-current electricity. Magnetostatics. Quasi-stationary phenomena

55. The plates of a spherical capacitor, which are separated by a conducting medium with a conductivity σ^*, have potentials φ_1 and φ_2. Calculate the current across the capacitor and the resistance of the spherical layer. The radiuses of the plates are r_1 and r_2.

56. Find the law of refraction of the lines of current at a flat boundary between two conducting media with conductivities σ_1^* and σ_2^*.

57. The potential difference between two flat electrodes is V and the distance between them is d. The field-induced emission of electrons from one of the electrodes continues until a space charge forms between the electrodes, which opposes the external field. Find the relationship between the current density and the potential difference applied to the electrodes.

58. Find the magnetic field strength inside and outside a cylindrical conductor with an electric current whose density j is the same through any section of the conductor. The radius of the cylinder is R.

59. Find the magnetic field strength inside a cylindrical cavity in a cylindrical conductor with an electric current whose density j is the same through any section of the con-

ductor. The axes of the cavity and the conductor are parallel and separated by a distance a.

60. An electric current flows through an infinitely long conductor of radius R. The current density is a/ρ for $\rho \leqslant R$, where R is the radius of the conductor and ρ is the distance from the axis of the conductor. Find the vector potential and the magnetic field strength inside and outside the conductor.

61. Find the magnetic field strength of a plane with a surface current of density i that is the same in any point of the plane.

62. Surface currents with densities i flow along two parallel planes. Find the magnetic field strength when the currents flow (a) in the same direction and (b) in opposite directions.

63. Consider a strip of infinite length and width a made of a conducting material. A current with a surface density i flows uniformly through the strip. Find the magnetic field. Consider the limiting case as the width of the strip tends to infinity and compare the obtained result with that of Problem 61.

64. Electric currents I flow in opposite directions along two straight, parallel conductors of infinite length placed apart at a distance d. Determine the vector potential of the system.

65. Find the vector potential and the magnetic field strength created by a current I flowing along a ring of radius R. Examine the special case when the observation point lies on the axis of the ring.

66. Find the magnetic field strength and the vector of magnetic induction created by a uniformly magnetized ball. The radius of the ball is R and the magnetization vector is $\mathbf{M}$.

67. Determine the magnetic field strength on the axis of a magnet of cylindrical shape. The radius of the magnet is R, its length is d, and the magnetization is M_0.

68. Find the magnetic moment of an electrically charged ball that rotates uniformly with an angular frequency Ω. The charge e is uniformly distributed over the volume of the ball. Show that the gyromagnetic ratio for this system is $e/(2m)$, where m is the mass of the ball.

69. A sphere of radius R rotates about the z-axis with an angular velocity Ω. Its surface is electrically charged with a density σ. Find the vector potential and the magnetic field strength inside and outside the sphere.

70. Calculate the force between two straight parallel wires of infinite length with electric currents I_1 and I_2 if the distance between the wires is d. The permeability of the medium between the conductors is μ.

71. Calculate the force between two coaxial wire loops of radiuses R_1 and R_2 with electric currents flowing in the same direction. The distance between the centres of the loops is d, and both are placed in a medium with a permeability μ.

72. Find the self-inductance L per unit length of a transmission line that consists of two coaxial cylinders of radiuses R_1 and R_2 ($R_1 < R_2$). The space between the conductors is filled with a substance having a permeability μ.

73. A solid conductor of radius R_1 with a permeability μ_1 is placed inside a cylinder of radius R_2. The space between the conductor and the cylinder is filled with a substance having a permeability μ_2. Determine the self-inductance per unit length of this contour.

74. The diamagnetic susceptibility per unit volume is

$$\chi = -\frac{Ze^2N}{6m}\mu_0\overline{r^2}$$

where Ze is the electric charge of the atomic nucleus, e and m are the electron charge and electron mass, N is the number of atoms per unit volume, and $\overline{r^2}$ is the mean square of the radius of the atom, i.e.

$$\overline{r^2} = \frac{1}{Ze}\int r^2\rho(r)\,d\tau$$

where $\rho(r)$ is the charge density in the atom. Determine χ for atomic hydrogen. In this case

$$\rho(r) = \frac{e}{\pi a^3}e^{-2r/a}$$

where $a = 0.528 \times 10^{-10}$ m (the Bohr radius).

75. Show that for a constant and uniform magnetic field $\mathbf{B}$ the vector potential can be chosen in the form $\mathbf{A} = \frac{1}{2}[\mathbf{B} \times \mathbf{r}]$.

76. Find the distribution of the electric and magnetic fields inside a cylindrical conductor which carries an alternating current with frequency ω. The electrical conductivity of the conductor is σ^*.

77. Consider a specimen of magnetic material subjected to a constant magnetic field of intensity $\mathbf{H}_0$. Under the influence of this field a magnetization $\mathbf{M}_0$ parallel to the field appears inside the specimen. In addition to the constant field a variable magnetic field that is perpendicular to $\mathbf{H}_0$ and that rotates with an angular velocity ω is imposed on the specimen. The amplitude h of the intensity of this variable field satisfies the condition $h \ll H_0$. Determine the additional magnetization caused by the variable field and find the conditions for magnetic resonance.

78. A ball made of magnetic material is inserted in a constant magnetic field. The intensity of the field inside the ball is $\mathbf{H}_0$. If we assume the size of the ball to be considerably less than the wavelength of natural oscillations of the magnetic moment, what will be the frequency of these oscillations?

79. A sample of infinite length of nonconducting magnetic material is subjected to a constant magnetic field of intensity $\mathbf{H}_0$. Find the frequency of natural oscillations if the wavelength of these oscillations is considerably less than the wavelength of electromagnetic waves (the magnetostatic approximation).

80. Determine in the magnetostatic approximation the natural frequency of vibrations of a plate made of magnetic material and having a metallic covering. The thickness of the plate is d, the intensity of the constant magnetic field is H_0, the plate is magnetized normally to its surface, and the covering is ideal.

81. Determine the natural frequency of vibrations of a plate made of a magnetic material and magnetized normally to its surface. The plate is placed in a vacuum, its thickness is d, and the intensity of the constant magnetic field is H_0. Use the results of Problem 80.

82. Find the natural frequencies of two coupled circuits if the self-inductances are L_1 and L_2, the mutual inductance is L_{12}, the capacitances of the circuits are C_1 and C_2, and all resistances in the circuits are zero.

Propagation of electromagnetic waves. Wave guides. Resonators. Magnetohydrodynamics

83. Two plane, monochromatic waves are linearly-polarized in perpendicular directions. Determine the polarization of the resulting wave if both waves propagate in the same direction, their frequencies are the same, the amplitudes of the waves are E_{01} and E_{02} respectively, and the phase difference for the waves is φ.

84. Determine the damping of electromagnetic radiation in a medium in total internal reflection.

85. A plane-polarized wave strikes the surface of a nonmagnetic material normally to this surface. The permittivity of the material is ε and its electrical conductivity is σ^*. Find the reflectance R. Consider the limiting case of an ideal conductor.

86. Determine the amplitudes of a reflected and a refracted wave in a plane-parallel plate. The thickness of the plate is d and its permittivity is ε. Find the conditions for the reflection of electromagnetic waves from the plate to be minimal.

87. A surface wave whose magnetic field strength is perpendicular to its line of propagation (a TM wave) travels along the interface of two dielectrics. The permittivities of the dielectrics are opposite in sign and equal to ε_1 and $-|\varepsilon_2|$. Find the dispersion equation.

88. An electromagnetic wave falls onto a plane surface of a semi-infinite crystal at an angle θ_1. Determine the directions of propagation of the ordinary and extraordinary rays in the crystal. The optic axis of the crystal is perpendicular to its surface.

89. Assume that a medium consists of elastically coupled charged particles and that the force constants are different for mutually perpendicular directions. Find the permittivity tensor of the medium if the volume concentration of the particles is N.

90. Show that if $\varepsilon(\omega)$ is an analytic function in the upper half of the complex ω plane and approaches unity as $|\omega|$ tends to infinity, the real and imaginary parts of

$\varepsilon(\omega)$ satisfy the following equations:

$$\operatorname{Re}\varepsilon(\omega)=1+\frac{1}{\pi}\,\mathrm{P}\int\limits_{-\infty}^{\infty}\frac{\operatorname{Im}\varepsilon(\omega')}{\omega'-\omega}\,d\omega'$$

$$\operatorname{Im}\varepsilon(\omega)=-\frac{1}{\pi}\,\mathrm{P}\int\limits_{-\infty}^{\infty}\frac{\operatorname{Re}\varepsilon(\omega')-1}{\omega'-\omega}\,d\omega'$$

where P means the principal value of the integral.

91. A medium consisting of elastically coupled electrons has a permittivity equal to

$$\varepsilon(\omega)=1+\frac{Ne^2}{m\varepsilon_0}\sum_k\frac{f_k}{\omega_k^2-\omega^2-i\gamma_k\omega}$$

where N is the volume concentration of the electrons, e and m are the electron charge and mass respectively, and f_k and ω_k are constants. Show that this equality satisfies the dispersion relations in Problem 90.

92. Assume that a substance consisting of quasi-elastically coupled electrons of volume concentration N is placed in a homogeneous magnetic field with an induction B_0. A linearly-polarized monochromatic light wave whose wave vector is parallel to the magnetic field falls onto the substance. Find the angle of rotation of the plane of polarization of the wave when it has travelled a distance l in the substance.

93. A system of anharmonic charged oscillators is placed in an alternating electromagnetic field. The potential energy of an oscillator as a function of displacement is given by the relation

$$U=\frac{1}{2}kr^2+\frac{1}{3}\sum_{i,\,j,\,l=1}^{3}\beta_{ijl}x_ix_jx_l$$

Assuming that the factors β_{ijl} are small, find the polarization vector up to terms linear in β_{ijl} and quadratic in the electric field vector. Show that the polarization vector has terms corresponding to oscillations with a frequency twice the one of the incident wave. The mass of an oscillator is m, its natural frequency is ω_0, its electric charge is e, the volume

concentration of oscillators is N, and the frequency of the external field is ω.

94. Find the nonlinear polarization of the system of oscillators of Problem 93 if the external field consists of two monochromatic waves with frequencies ω_1 and ω_2.

95. Assume that a medium consists of parallel chains of evenly distributed oscillators with natural frequencies ω_0. The neighbouring oscillators in a chain interact via the Hooke's law. Examine the propagation of an electromagnetic wave along the chains when the wavelength is much greater than the distance between the oscillators. Find the index of refraction for such waves.

96. Examine the propagation of electromagnetic waves in the space between two conducting plates separated by a dielectric medium. The distance between the plates is d and the permittivity of the medium is ε.

97. Find the intensity, dispersion equation, and frequency limit for the TE and TM waves in a rectangular wave guide with ideally conducting walls. The dimensions of the wave guide are a and b.

98. What is the relation between the tangential components of an electric and a magnetic field near a conductor?

99. Determine the damping of TM waves in a rectangular wave guide with dimensions a and b. The conductivity of the walls is σ^* and their permeability is μ.

100. Find the dispersion equation for electromagnetic waves in a cylindrical wave guide with ideally conducting walls. The radius of the guide is R.

101. Examine the propagation of electromagnetic waves along a cylindrical wave guide made of a dielectrical material with permittivity ε. The radius of the guide is R.

102. Examine the propagation of electromagnetic waves in a dielectric medium that fills the space between two conducting coaxial cylinders. The cylinders have radiuses R_1 and R_2.

103. Determine the electric field vector and the natural frequencies of TM waves in a rectangular cavity resonator. The dimensions of the resonator are a, b, and c, and its walls are ideal conductors.

104. Show that the number of oscillations in a frequency interval $\Delta\omega$ is $\frac{L_1L_2L_3\omega^2\Delta\omega}{\pi^2c^3}$ for a rectangular resonator of

dimensions L_1, L_2, and L_3 and with ideally conducting walls.

105. Determine the electromagnetic field and natural frequencies of electromagnetic waves in a cylindrical resonator of radius R. The distance between the ends of the resonator is d.

106. A viscous, conducting, and incompressible liquid moves between two parallel infinite planes. A constant magnetic field $\mathbf{H}_0$ is directed perpendicularly to the planes. Determine the distribution of velocities in the liquid if it is in stationary flow. The distance between the planes is d.

107. A viscous, conducting, and incompressible liquid is placed between two conducting planes $z = 0$ and $z = d$. The plane $z = d$ moves along the x-axis with a velocity v_0. A uniform magnetic field $\mathbf{H}_0$ is directed along the z-axis, and an electric field $\mathbf{E}_0$ along the y-axis. Determine the distribution of velocities in the liquid.

108. An ionized plasma consists of ions and electrons. Assume the charge density fluctuations to be small. Examine the variation of charge density (concentration) and find the plasma frequency if this frequency is so large that the ions are not able to catch up with the field and stay fixed. Magnetic interactions and pressure gradients are negligible.

The radiation and scattering of electromagnetic waves

109. Obtain the equations for the potentials created by charges and currents in a vacuum provided div $\mathbf{A} = 0$ (the Coulomb gauge condition).

110. An electric current of density

$$\mathbf{j}(\mathbf{r},\ t) = I \sin\left(\frac{kd}{2} - k\,|\,z\,|\right)\delta(x)\,\delta(y)\,\mathbf{e}_3 e^{-i\omega t}$$

is generated in a thin linear antenna. Find the time average (over the period) of the antenna's intensity of the radiation in a unit solid angle. Consider the particular case when the antenna length d is several half-waves. Vector $\mathbf{e}_3$ is directed along the antenna.

111. Find the total radiation intensity of a linear antenna with a wave traveling from point $z = -l/2$ to point $z =$

$= l/2$, where it is absorbed completely, i.e. without reflection. The electric current in the wave is $I_0 \cos(\omega t - kz)$.

112. Examine the propagation of plane waves inside an isotropic and homogeneous dielectric, assuming that each volume element radiates a spherical wave which propagates with a velocity c. Determine the velocity of the plane waves in the medium.

113. A plane electromagnetic wave falls on a conducting cylinder of radius R. The cylinder's axis is perpendicular to the wave vector and parallel to the magnetic field. Find the electric and magnetic fields in the scattered wave. Calculate the energy flux scattered by a unit length of the cylinder.

114. Show that when two identical particles collide, there is no dipole radiation.

115. Find the intensity of radiation of a particle of mass m moving in a circular orbit of radius a under Coulomb forces. Express the answer in terms of the particle's energy.

·**116.** Determine the time it will take a particle moving in a circular orbit to "fall" on a charged centre because of the energy loss through radiation.

117. An atom radiates electromagnetic waves and stays in an excited state during time τ. The time dependence of the atom's electric field is

$$E(t) = E_0 e^{-\frac{t}{\tau} - i\omega_0 t}$$

Determine the width of the line radiated by the atom.

118. Find the differential cross section for the scattering of an elliptically polarized wave of frequency ω by an oscillator. The natural frequency of the oscillator is ω_0, its mass is m, its charge is e, and its damping coefficient is γ.

119. Examine the electromagnetic field created by the plane $z = 0$ with a surface current of density

$$\mathbf{i} = \mathbf{i}_0 e^{iq_x x + iq_y y - i\omega t}.$$

120. Consider a dipole with moment $\mathbf{p}$ that vibrates with a frequency ω in the origin of a coordinate system. A particle with a polarizability β is placed at a point with

a radius vector **d** (**d** $\perp$ **p**). Find the radiation intensity of electromagnetic waves for the system provided $d \ll \lambda$, where λ is the radiation's wavelength.

121. A dipole of moment **p** is located at a distance d from an infinite conducting plane and vibrates with a frequency ω. Determine the damping of the vibration if $d \ll \lambda$, where λ is the wavelength of the system's radiation. The conductivity of the material of the plane is σ^*.

122. A dipole of moment **p** vibrates with a frequency ω in a dielectric medium (of permittivity ε_1) at a distance d from the flat interface with another dielectric medium (of permittivity ε_2). Find the electric and magnetic fields of the waves radiated by the dipole. In all calculations $d \gg \lambda$.

123. A particle of charge e moves with a velocity **v** and then elastically bounces off a plane. Determine the long-wave part of the radiation spectrum at the moment of impact.

Special theory of relativity. Relativistic electrodynamics

124. Show that two successive Lorentz transformations in the same direction commute and are equivalent to one Lorentz transformation.

125. A plane electromagnetic wave propagates in a medium that moves with a velocity **v** relative to a reference frame K. Find the velocity of the wave in frame K if the index of refraction of the medium is n. Examine the case when $v \ll c$.

126. Find the approximate relationship between the energy of a slow particle and its momentum up to members proportional to $\left(\frac{p^2}{m^2c^2}\right)^2$. For slow particles $p^2 \ll m^2c^2$.

127. Find the path of a charged particle moving in a uniform electric field $\mathbf{E}_0$. Examine the limiting case of a slow particle.

128. Find the path of a charged particle moving in a uniform magnetic field $\mathbf{H}_0$.

129. Find the path of a relativistic particle of charge e_1 and mass m in the field of a fixed point charge e_2.

130. A particle of mass M decays into two particles of

masses m_1 and m_2. Find the energies of the decay products in the centre-of-mass system.

131. Find the kinetic energies of a μ-meson (rest energy 105.7 MeV) and a neutrino that were produced in the decay of a fixed π-meson (rest energy 139.6 MeV). Use the results of Problem 130.

132. Find the Lorentz equations for transforming the energy and the components of momentum of a particle from one reference frame to another frame moving with a velocity v relative to the first.

133. A particle moving with a velocity v decays into two particles whose energies in the centre-of-mass system are E_1 and E_2. Find the relationship between the angle of emergence and the energies of the products of decay in the laboratory frame of reference.

134. Find the relationship between the directions of the velocity of a particle in two reference frames that move with a relative velocity v.

135. Two particles with rest masses m_1 and m_2 and energies E_1 and E_2 collide elastically. If before the collision the second particle is at rest find the scattering angles in the laboratory frame, as functions of the energies E_1' and E_2' after the collision.

136. Determine the relationship between the frequency of a photon scattered by a stationary free electron and the scattering angle (the Compton effect).

137. Find the frequency of the photon emitted by an atom in an excited state. The excitation energy of the atom is ΔE, its mass is m, and the atom is stationary.

138. Show that the annihilation of an electron-positron pair into one photon is prohibited by the conservation law of 4-momentum.

139. Consider a mirror that moves with a velocity v in the direction opposite to its normal. A ray of light falls on the mirror at an angle θ. Determine the angle of reflection and the frequency shift of the reflected wave.

140. Prove that if vectors **E** and **H** are perpendicular to each other in one inertial reference frame, they are perpendicular in any other inertial frame.

141. Find an inertial reference frame in which the electric and magnetic fields are parallel.

142. Find the potentials of a uniformly moving charge by using the relativistic transformation of the static Coulomb field.

143. Find the Lorentz equations for transforming the components of the energy-momentum tensor.

144. Find the Lorentz equations for transforming the components of the electric and magnetic fields to a reference frame that moves with a velocity v.

145. Show that the wave equation is not invariant under the Galilean transformations but is invariant under the Lorentz transformations.

146. Show that $\mathbf{E}^2 - \mathbf{H}^2$ is invariant under the Lorentz transformations.

147. Show that if a magnetic moment $\boldsymbol{\mu}$ moves with a velocity v $(v \ll c)$, this gives rise to an electric dipole moment $\mathbf{p} = \frac{1}{c^2}[\mathbf{v} \times \boldsymbol{\mu}]$.

SECTION III

Quantum Mechanics

In quantum mechanics a state of a microparticle subjected to a potential field $V(\mathbf{r}, t)$ is described by a complex-valued wave function $\Psi(\mathbf{r}, t)$ which is determined from the time-dependent Schrödinger equation

$$i\hbar \frac{\partial \Psi(\mathbf{r}, t)}{\partial t} = -\frac{\hbar^2}{2m} \Delta \Psi(\mathbf{r}, t) + V(\mathbf{r}, t)\Psi(\mathbf{r}, t) \quad \text{(III-1)}$$

where m is the particle's mass, and $\hbar$ is the Planck constant h divided by 2π. If we multiply equation (III-1) by the complex conjugate function $\Psi^*(\mathbf{r}, t)$ and an equation of type (III-1) for $\Psi^*(\mathbf{r}, t)$ by $\Psi(\mathbf{r}, t)$, and then subtract the second product from the first, we get the so-called equation of continuity

$$\frac{\partial}{\partial t} |\Psi(\mathbf{r}, t)|^2 + \operatorname{div}\left[\frac{i\hbar}{2m}(\Psi \operatorname{grad} \Psi^* - \Psi^* \operatorname{grad} \Psi)\right] = 0 \quad \text{(III-2)}$$

This equation is the differential form of the conservation law for the quantity $\int |\Psi|^2 d\tau$, which makes it possible to interpret $|\Psi(\mathbf{r}, t)|^2$ as the probability density of finding the particle at point $\mathbf{r}$ at time t, so $|\Psi(\mathbf{r}, t)|^2 d\tau$ is the probability of finding the particle under consideration in the volume element $d\tau$. Then vector

$$\mathbf{j} = \frac{i\hbar}{2m}(\Psi \operatorname{grad} \Psi^* - \Psi^* \operatorname{grad} \Psi) \quad \text{(III-3)}$$

is the probability current density.

Such an interpretation of $|\Psi|^2$ leads us to the conditions which $\Psi(\mathbf{r}, t)$ must satisfy: the wave function must be single-valued, finite in every point of space, and continuous together with all its first derivatives. This last requirement

(the continuity of the first derivatives) is violated only at points where the potential has a discontinuity of the second kind.

If the potential does not depend explicitly on time, there exists the so-called stationary solution of the Schrödinger equation

$$\Psi(\mathbf{r},\ t) = \psi(\mathbf{r})\, e^{-\frac{i}{\hbar}Et}$$

for which $\rho = |\Psi|^2$ and $\mathbf{j}$ are time-independent. Substituting this solution into equation (III-1), we come to the time-independent Schrödinger equation

$$\left[-\frac{\hbar^2}{2m}\Delta + V(\mathbf{r})\right]\psi(\mathbf{r}) = E\psi(\mathbf{r}) \qquad \text{(III-4)}$$

Generally speaking, the solutions of equation (III-4), which must satisfy the conditions mentioned above, exist only for certain values of parameter E, i.e. for certain values of the energy of the microparticle. Sometimes there is a whole range of values of E for which ψ is finite, single-valued, and continuous (the case of a continuous spectrum of E). But sometimes such a solution exists only for a discrete spectrum of E. In still other cases the two spectra exist simultaneously.

For a discrete spectrum the wave function ψ can be normalized to unity:

$$\int |\psi|^2\, d\tau = 1 \qquad \text{(III-5)}$$

The eigenfunctions for states with different energies are mutually orthogonal.

For a continuous spectrum the orthonormality condition can be written with the help of the Dirac delta function:

$$\int \psi^*(\mathbf{r},\ E)\,\psi(\mathbf{r},\ E')\, d\tau = \delta(E - E') \qquad \text{(III-6)}$$

where the δ-function, as always, is defined from the condition that for an arbitrary function $f(x)$

$$\int_{-\infty}^{\infty} f(x)\,\delta(x-a)\, dx = f(a)$$

In a variety of problems the solution of the time-independent Schrödinger equation is sought in the form $\psi = e^{iS(\mathbf{r})/\hbar}$

(the Wentzel-Kramers-Brillouin approximation, usually called the WKB approximation), with S (**r**) satisfying the equation

$$\frac{1}{2m}(\text{grad}\, S)^2 + \frac{i\hbar}{2m}\Delta S + V(\mathbf{r}) = E \qquad \text{(III-7)}$$

By expanding S (**r**) in a series of powers of $i\hbar$

$$S = S^0 + i\hbar S' + \ldots$$

we find, in the zeroth-order approximation, the time-independent Hamilton-Jacobi equation for S^0, and the first- and higher-order approximations give us small corrections to S^0.

If the variables are separable, the solution of the Hamilton-Jacobi equation is of the form

$$S^0 = \sum_i S_i^0(q_i), \quad \text{and} \quad S_i^0(q_i) = \int p_i\, dq_i$$

where p_i is the canonical momentum conjugate to the generalized coordinate q_i. In particular, for the one-dimensional case,

$$S^0 = \int \sqrt{2m[E - V(x)]}\, dx \qquad \text{(III-8)}$$

If V (x) is a one-dimensional potential barrier, i.e. $V(x) > E$ in the region $a \leqslant x \leqslant b$, inside the barrier S^0 is imaginary, and so ψ acquires the factor

$$\exp\left[-\frac{\sqrt{2m}}{\hbar}\int_a^b \sqrt{V(x) - E}\, dx\right]$$

If we define the transmittance of the barrier as the ratio of the probability current densities of the transmitted wave and the incident wave and use the WKB method, we can get the expression

$$D = \left|\frac{j_{\text{trans}}}{j_{\text{in}}}\right| = D_0 \exp\left[-2\frac{\sqrt{2m}}{\hbar}\int_a^b \sqrt{V(x) - E}\, dx\right] \qquad \text{(III-9)}$$

where D_0 is a constant.

If a microparticle is in periodic motion in a potential well, the condition that ψ be a single-valued function in

the zeroth-order WKB approximation brings us to the following formula:

$$\oint p_i \, dq_i = n_i h \tag{III-10}$$

where $n_i = 0, 1, 2, \ldots$.

These have come to be known as the Bohr-Sommerfeld quantization rule. It was Sommerfeld who in 1916 formulated the postulates of Bohr's theory of spectra in a way that made it possible to select allowed orbits from the continuum of classically possible orbits. The integral in (III-10) is taken along a closed orbit.

According to Bohr, the electrons move in orbits restricted by condition (III-10) and do not radiate in spite of their acceleration. Radiation is emitted or absorbed when an electron makes a discontinuous transition from one allowed orbit to another, and the frequency of this radiation is

$$\omega_{mn} = (E_m - E_n)/\hbar \tag{III-11}$$

where E_m and E_n are the corresponding energies of the electron on the orbits.

If we define the expectation value of the particle's momentum via the probability current density as $\langle \mathbf{p} \rangle = \int m\mathbf{j} \, d\tau$, then substitute expression (III-3) for $\mathbf{j}$, and integrate the second member in the right-hand side by parts, we get

$$\langle \mathbf{p} \rangle = \int \Psi^* (-i\hbar\nabla) \Psi \, d\tau \tag{III-12}$$

where we have used the fact that $|\Psi|^2 = 0$ on the boundary of the integration range. In a similar manner we can define the expectation value of the particle's position as $\langle x \rangle = \int \Psi^* x \Psi \, d\tau$.

Operators. In quantum mechanics the momentum of classical physics is represented by an operator $\hat{\mathbf{p}} = -i\hbar\nabla$, and position x by an operator $\hat{x} = x$,

$$\langle \mathbf{p} \rangle = \int \Psi^* \hat{\mathbf{p}} \Psi \, d\tau \quad \text{and} \quad \langle x \rangle = \int \Psi^* \hat{x} \Psi \, d\tau$$

In the case of an operator representing a physical observable $F(\mathbf{p}, \mathbf{r})$ which is a function of both momentum and

position, we construct the same function but with operators $\hat{\mathbf{r}}$ and $\hat{\mathbf{p}}$ instead of $\mathbf{r}$ and $\mathbf{p}$:

$$\hat{F}(\mathbf{p}, \mathbf{r}) = F(\hat{\mathbf{p}}, \hat{\mathbf{r}}) = F(-i\hbar\nabla, \mathbf{r}) \qquad \text{(III-13)}$$

For a series of measurements of an observable F, the expectation value of this observable in a state $\Psi(\mathbf{r}, t)$ is

$$\langle F\rangle = \int \Psi^*(\mathbf{r}, t)\, \hat{F}\Psi(\mathbf{r}, t)\, d\tau \qquad \text{(III-14)}$$

where the wave function $\Psi(\mathbf{r}, t)$ is a vector in a Hilbert space and satisfies the orthonormality condition

$$\int \Psi^*(\mathbf{r}, t)\, \Psi(\mathbf{r}, t)\, d\tau = 1$$

The hermitian conjugate of the operator $\hat{F}$, denoted $\hat{F}^+$, is defined by the relation

$$\int \Psi_1^* \hat{F}\Psi_2\, d\tau = \int \Psi_2 (\hat{F}^+\Psi_1)^*\, d\tau \qquad \text{(III-15)}$$

where Ψ_1 and Ψ_2 are any two functions from a Hilbert space.

The operator representing an observable must be hermitian (self-conjugate) and linear, i.e. satisfy the following conditions

$$\hat{F}^+ = \hat{F} \qquad \text{(III-16)}$$

$$\hat{F} \sum_k c_k \Psi_k = \sum_k c_k \hat{F}\Psi_k \qquad \text{(III-17)}$$

For the case when F is the product of two observables A and B, $F = AB$, condition (III-16) is satisfied only if $\hat{A}\hat{B} = \hat{B}\hat{A}$. But if the two operators $\hat{A}$ and $\hat{B}$ do not commute, the observable F is represented by the operator $\hat{F} = \frac{1}{2}(\hat{A}\hat{B} + \hat{B}\hat{A})$.

According to definition (III-13), we can introduce the following energy operators: the kinetic energy operator $\hat{T} = \frac{\hat{\mathbf{p}}^2}{2m}$, the potential energy operator $\hat{V}$, and the total energy operator (the Hamiltonian) $\hat{H}$. Explicitly,

$$\hat{T} = -\frac{\hbar^2}{2m}\Delta, \quad \hat{V} = V(\mathbf{r}), \quad \hat{H} = \hat{T} + \hat{V} = -\frac{\hbar^2}{2m}\Delta + V(\mathbf{r}) \qquad \text{(III-18)}$$

The three components of the angular momentum and its square are represented by the operators

$$\hat{L}_x = \hat{y}\hat{p}_z - \hat{z}\hat{p}_y, \quad \hat{L}_y = \hat{z}\hat{p}_x - \hat{x}\hat{p}_z, \quad \hat{L}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x,$$

$$\hat{\mathbf{L}}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2 \tag{III-19}$$

Let us write $\hat{L}_z$ and $\hat{\mathbf{L}}^2$ in spherical coordinates. We have

$$\hat{L}_z = -i\hbar \frac{\partial}{\partial \varphi}$$

$$\hat{\mathbf{L}}^2 = -\hbar^2 \Delta_{\theta\varphi} = -\hbar^2 \left[\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial}{\partial\theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \right] \tag{III-20}$$

If we define the variance of the distribution of λ about the expectation value $\langle \lambda \rangle$ as

$$\langle \Delta\lambda^2 \rangle = \langle (\lambda - \langle \lambda \rangle)^2 \rangle = \int \psi^* (\hat{L} - \langle \lambda \rangle)^2 \psi \, d\tau$$

and then use the relationship (III-15) for $\hat{L} - \langle \lambda \rangle$, we can show that

$$\langle \Delta\lambda^2 \rangle = \int |(\hat{L} - \langle \lambda \rangle) \psi|^2 \, d\tau$$

Thus, if the variance of the distribution of λ in a state with the wave function ψ is zero, i.e. if $\langle \Delta\lambda^2 \rangle = 0$, the condition

$$\hat{L}\psi = \langle \lambda \rangle \psi \tag{III-21}$$

holds true.

Hence, the state with the wave function must be described by an eigenfunction of $\hat{L}$ which satisfies condition (III-21) and is finite, single-valued, and continuous.

The scalars λ_n, for which ψ_n satisfy the above mentioned conditions, are called eigenvalues of $\hat{L}$. A specific eigenvalue λ_n is nondegenerate or α-fold degenerate depending on whether one or α linearly independent eigenfunctions correspond to it.

The capability of operators to commute has important physical significance. If $\hat{L}\hat{M} - \hat{M}\hat{L} = 0$, i.e. if the result of $\hat{L}\hat{M} - \hat{M}\hat{L}$ acting on an arbitrary function ψ is zero, operators $\hat{L}$ and $\hat{M}$ have common eigenvalues and, hence, the corresponding observables λ and μ can simultaneously

assume exact values. Otherwise, if $\hat{L}\hat{M} - \hat{M}\hat{L} \neq 0$, we have the uncertainty relations for λ and μ.

For example, if a particle is in a spherically symmetric field $V(r)$, the Hamiltonian $\hat{H}$ commutes with $\hat{\mathbf{L}}^2$ and $\hat{L}_z$, and its eigenfunctions can be sought in the form

$$\psi(r, \theta, \varphi) = R(r)\, Y(\theta, \varphi) \tag{III-22}$$

where Y satisfies the following equations:

$$-\hbar^2 \Delta_{\theta\varphi} Y = \lambda Y \quad \text{and} \quad -i\hbar \frac{\partial Y}{\partial \varphi} = aY$$

(It is easy to see that $\hat{\mathbf{L}}^2$ and $\hat{L}_z$ always commute.) The solutions of these equations that satisfy all the conditions for eigenfunctions are the spherical harmonics

$$Y_{lm}(\theta, \varphi) = P_{lm}(\cos\theta)\, e^{im\varphi} \tag{III-23}$$

From this it follows that $\lambda = \hbar^2 l(l+1)$ and $a = \hbar m$, where $l = 0, 1, 2, \ldots$ and $m = 0, \pm 1, \pm 2, \ldots, \pm l$; $P_{lm}(x)$ are the associated Legendre polynomials of degree l and order m.

For an electron in the Coulomb field of a nucleus with a charge Ze, the radial part of function (III-22) assumes the form

$$R_{nl}(\rho) = e^{-\rho/n}\rho^l \sum_{k=0}^{n-l-1} b_k \rho^k \tag{III-24}$$

where the coefficients b_k satisfy the recurrence relations

$$b_{k+1} = b_k \frac{2}{n} \frac{k+l+1-n}{(k+l)(k+l+1) - l(l+1)} \tag{III-24a}$$

Here the dimensionless variable $\rho = \frac{r}{a}$, where $a = \frac{\hbar^2}{Z\mu e^2}$, and l assumes the values $0, 1, \ldots, n-1$.

Each eigenvalue of the energy of the electron $E_n = -\frac{Z^2\mu e^4}{2\hbar^2 n^2}$ has corresponding to it n^2 eigenfunctions of type (III-22), bearing in mind (III-23) and (III-24), $n = 1, 2, \ldots$.

Integrals of motion. If we find the time derivative of expression (III-14) and use the time-dependent Schrödinger equation $\hat{H}\Psi = i\hbar \frac{\partial \Psi}{\partial t}$, we can show that the time derivative

of an observable λ has corresponding to it the operator

$$\frac{d\hat{L}}{dt}=\frac{\partial\hat{L}}{\partial t}+\frac{i}{\hbar}(\hat{H}\hat{L}-\hat{L}\hat{H}) \tag{III-25}$$

When $\frac{d\hat{L}}{dt}=0$, λ is an integral of motion. If $\hat{H}$ and $\hat{L}$ commute, $\hat{L}$ does not depend explicitly on time.

Applying equation (III-25), we can find the quantum mechanical laws of motion, i.e. the operators $\frac{d\hat{\mathbf{p}}_i}{dt}$ and $\frac{d\hat{x}_i}{dt}$.

The eigenfunctions of a hermitian operator form a complete and orthonormalized set of functions, i.e.

$$\int \psi_m^*(\mathbf{r})\,\psi_n(\mathbf{r})\,d\tau=\delta_{mn}$$

for a discrete spectrum of eigenvalues,

$$\int \psi^*(\mathbf{r},\,\lambda)\,\psi(\mathbf{r},\,\lambda')\,d\tau=\delta(\lambda-\lambda')$$

for a continuous spectrum, and an arbitrary wave function can be expanded in terms of the eigenfunctions of $\hat{L}$:

$$\psi(\mathbf{r})=\sum_n c_n\psi_n(\mathbf{r})+\int_{\lambda_1}^{\lambda_2} c(\lambda)\,\psi(\mathbf{r},\,\lambda)\,d\lambda \tag{III-26}$$

If we use the above-mentioned orthonormality conditions, we find that

$$c_n=\int \psi_n^*(\mathbf{r})\,\psi(\mathbf{r})\,d\tau \quad \text{and} \quad c(\lambda)=\int \psi^*(\mathbf{r},\,\lambda)\,\psi(\mathbf{r})\,d\tau \tag{III-27}$$

where

$$\sum_n |c_n|^2+\int_{\lambda_1}^{\lambda_2} |c(\lambda)|^2\,d\lambda=1$$

From the equation

$$\langle\lambda\rangle=\sum_n |c_n|^2\lambda_n+\int_{\lambda_1}^{\lambda_2} \lambda\,|c(\lambda)|^2\,d\lambda \tag{III-28}$$

it follows that $|c_n|^2$ and $|c(\lambda)|^2\,d\lambda$ are, respectively, the probabilities of finding in state $\psi(\mathbf{r})$ the eigenvalue $\lambda=\lambda_n$

(in the discrete spectrum) or the eigenvalue in the limits from λ to $\lambda + d\lambda$ (in the continuous spectrum).

Theory of representations. Matrices. Expansion (III-28) replaces the wave function $\psi(\mathbf{r})$ by the coefficients c_n or $c(\lambda)$, and value x by value λ, i.e. there is a transition to a new representation, the λ-representation. If in the coordinate representation the operator $\hat{M}$ links functions φ and ψ,

$$\varphi(\mathbf{r}) = \hat{M}\psi(\mathbf{r})$$

and if $\varphi(\mathbf{r}) = \sum_n b_n\psi_n$ and $\psi(\mathbf{r}) = \sum_n c_n\psi_n$, we find that in the λ-representation

$$b_k = \sum_n \langle k \mid M \mid n\rangle c_n$$

where $\langle k \mid M \mid n\rangle = \int \psi_k^*\hat{M}\psi_n\, d\tau$ are the matrix elements of $\hat{M}$. In the transition to the λ-representation a matrix has replaced the operator.

An operator $\hat{L}$ in its own representation reduces to a diagonal matrix since

$$\langle k \mid L \mid n\rangle = \int \psi_k^*\hat{L}\psi_n\, d\tau = \lambda_n\, \delta_{kn}$$

with the diagonal elements as the eigenvalues. This holds true for an operator with a discrete spectrum of eigenvalues. An operator with a continuous spectrum in its own representation reduces to a multiplier equal to the independent variable (in the momentum representation $\hat{\mathbf{p}}$ equals $\mathbf{p}$). It is evident that for a hermitian operator $\hat{M}$ the matrix elements satisfy the following relationship:

$$\langle k \mid \hat{M} \mid n\rangle = \langle n \mid \hat{M} \mid k\rangle^*$$

A particle in an electromagnetic field. In order to generalize the Schrödinger equation for the motion of a particle in an electromagnetic field specified by an electric field vector $\mathbf{E}$ and a vector of magnetic induction $\mathbf{B}$, which, in turn, can be defined via potentials $\mathbf{A}(\mathbf{r}, t)$ and $\varphi(\mathbf{r}, t)$ as

$$\mathbf{B} = \operatorname{curl} \mathbf{A}, \quad \mathbf{E} = -\operatorname{grad} \varphi - \frac{\partial \mathbf{A}}{\partial t},$$

we must replace the momentum operator in the Hamiltonian $\hat{H}$ by the operator $\hat{\mathbf{p}} - e\mathbf{A}$ and write the potential ener-

gy as $e\varphi$, i.e. in the presence of an electromagnetic field

$$\hat{H}=\frac{1}{2m}(\hat{\mathbf{p}}-e\mathbf{A})^2+e\varphi$$

(This agrees with the classical case of a particle in a magnetic field.) Choosing the vector potential so that div $\mathbf{A}=0$ and neglecting the term $e^2\mathbf{A}^2$ as being small, we find the wave equation of quantum mechanics:

$$-\frac{\hbar^2}{2m}\Delta\Psi+i\hbar\frac{e}{m}(\mathbf{A}\cdot\operatorname{grad}\Psi)+e\varphi\Psi=i\hbar\frac{\partial\Psi}{\partial t} \quad \text{(III-29)}$$

Spin. The experiments of Stern and Gerlach and also the doubling of the energy levels E_{nl} of an electron in an atom show that apart from having an orbital angular momentum $\mathbf{L}=[\mathbf{r}\times\mathbf{p}]$ (we have mentioned this fact before), it has a spin angular momentum, or simply spin. The components of this spin momentum, S_x, S_y, and S_z, can assume only two distinct values, $+\hbar/2$ and $-\hbar/2$.

If we introduce the dimensionless spin operators $\hat{\sigma}_x$, $\hat{\sigma}_y$, $\hat{\sigma}_z$ so that $\hat{S}_x=\frac{\hbar}{2}\hat{\sigma}_x$, $\hat{S}_y=\frac{\hbar}{2}\hat{\sigma}_y$, $\hat{S}_z=\frac{\hbar}{2}\hat{\sigma}_z$ and in such a way that the operators satisfy the anticommutation relations $\hat{\sigma}_x\hat{\sigma}_y=-\hat{\sigma}_y\hat{\sigma}_x=i\hat{\sigma}_z$, (etc. cyclic), we can represent these operators by the so-called Pauli matrices. In the σ_z-representation

$$\hat{\sigma}_x=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad \hat{\sigma}_y=\begin{pmatrix}0&-i\\i&0\end{pmatrix},\quad \hat{\sigma}_z=\begin{pmatrix}1&0\\0&-1\end{pmatrix} \quad \text{(III-30)}$$

The wave function will then depend on the spin variable $\sigma=\pm1$. This dependence is made explicit by using a two-component matrix

$$\psi(\mathbf{r},\sigma)=\begin{pmatrix}\psi_1(\mathbf{r})\\\psi_2(\mathbf{r})\end{pmatrix}$$

(the spin variable assumes only two values). The two quantities $|\psi_1(\mathbf{r})|^2$ and $|\psi_2(\mathbf{r})|^2$ are interpreted as the probability densities of finding the electron at point $\mathbf{r}$, with $\sigma=+1$ (called "spin up") or $\sigma=-1$ (called "spin down"), i.e. with $S_z=+\hbar/2$ or $S_z=-\hbar/2$.

If it is possible to neglect the coupling between the electron's orbital angular momentum and the spin, we can

separate the position and spin variables, $\psi(\mathbf{r}, \sigma) = \psi_0(\mathbf{r})\chi(\sigma)$.

Since the electron has an intrinsic magnetic dipole moment by virtue of its spin, the operator for this moment being $\hat{\boldsymbol{\mu}} = \frac{e\hbar}{2m}\hat{\boldsymbol{\sigma}}$, the electron interacts with any external magnetic field, and the energy operator of this interaction is $-(\boldsymbol{\mu}\cdot\mathbf{B})$, where $\mathbf{B}$ is the vector of magnetic induction of the field. We thus come to the Pauli equation, which accounts for the electron spin:

$$\frac{1}{2m}(\hat{\mathbf{p}}-e\mathbf{A})^2\Psi(\mathbf{r}, \sigma, t)-(\hat{\boldsymbol{\mu}}\cdot\mathbf{B})\Psi(\mathbf{r}, \sigma, t) + e\varphi\Psi(\mathbf{r}, \sigma, t) = i\hbar\frac{\partial\Psi(\mathbf{r}, \sigma, t)}{\partial t} \quad \text{(III-31)}$$

This equation can be made simpler if we write out the first term explicitly and proceed as we did in evaluating equation (III-29).

We can write the time-independent equation in the same way.

Approximate methods of solving quantum mechanical problems. When the equation for the eigenfunctions and eigenvalues of an operator (for instance, of the Hamiltonian) cannot be solved exactly, we can use approximate methods. One of these is the time-independent perturbation theory.

Suppose we know the eigenfunctions and the eigenvalues for a Hamiltonian $\hat{H}_0$:

$$\hat{H}_0\psi_n^0 = E_n^0\psi_n^0$$

and we ask for the eigenfunctions and eigenvalues for the Hamiltonian $\hat{H} = \hat{H}_0 + \hat{W}$:

$$\hat{H}\psi_k = E_k\psi_k$$

in the form of an expansion in terms of the eigenfunctions of $\hat{H}_0$, i.e.

$$\psi_k = \sum_n c_{kn}\psi_n^0$$

In the equation for the coefficients c_{kn}

$$c_{kl}(E_l^0 - E_k) + \sum_n \langle l | W | n\rangle c_{kn} = 0$$

we must express the quantities c_{kl} and E_k as power series in a small parameter that enters the perturbation energy operator: $\hat{W} = \varepsilon \hat{V}_1$, where ε is small. Using the method of successive approximations, we get the wave function and the energy level:

$$\psi_k = \psi_k^0 + \sum_{n \neq k} \frac{\langle n | W | k \rangle}{E_k^0 - E_n^0} \psi_n^0 + \dots$$

$$E_k = E_k^0 + \langle k | W | k \rangle + \sum_{n \neq k} \frac{|\langle n | W | k \rangle|^2}{E_k^0 - E_n^0} + \dots \quad \text{(III-32)}$$

provided the corresponding energy level of the unperturbed system is nondegenerate.

If the level E_k^0 is α-fold degenerate, the zeroth-order approximation of the wave function of the perturbed system is expressed as a linear combination of the degenerate eigenfunctions of $\hat{H}_0$, which correspond to the same energy level E_k^0:

$$\psi_k = \sum_{\beta=1}^{\alpha} c_\beta \psi_{k\beta}^0$$

where

$$\hat{H}_0 \psi_{k\beta}^0 = E_k^0 \psi_{k\beta}^0 \quad (\beta = 1, 2, \dots, \alpha)$$

For the parameters c_β we get a system of α homogeneous linear equations

$$(E_k^0 - E + \langle k_\beta | W | k_\beta \rangle) c_\beta + \sum_{\gamma \neq \beta} \langle k_\beta | W | k_\gamma \rangle c_\gamma = 0$$

where $\beta, \gamma = 1, 2, \dots, \alpha$. Nontrivial solutions for this system exist if the system determinant is zero, or

$$| (E_k^0 - E) \delta_{\beta\gamma} + \langle k_\beta | W | k_\gamma \rangle | = 0 \quad (\beta, \gamma = 1, 2, \dots, \alpha) \quad \text{(III-33)}$$

Generally speaking, this equation defines α values of E, and the foregoing equations make it possible to calculate the coefficients c_β and, hence, the wave functions ψ_k.

Another approximate method of solving the eigenvalue equation is the Ritz variational method, based on the variational principle of quantum mechanics. It can be shown that for the eigenfunction of $\hat{H}$ for the ground state, ψ, where $\hat{H}\psi = \lambda\psi$ and $\int |\psi|^2 \, d\tau = 1$, the functional $\langle H \rangle [\psi] =$

$= \int \psi^* \hat{H} \psi \, d\tau$ is either maximized or minimized. Whence, if we calculate $\langle H \rangle [\varphi] = \int \varphi^* \hat{H} \varphi \, d\tau$ for a trial function $\varphi = \varphi(\mathbf{r}, A, \alpha, \ldots)$ that satisfies the normalization condition $\int |\varphi|^2 \, d\tau = 1$, by adjusting parameters $A, \alpha, \ldots$ we can calculate the upper bound of $\langle H \rangle$ on E_0 (the ground-state energy).

To solve the eigenvalue problem for a system of n particles we must use the Schrödinger equation with a Hamiltonian

$$\hat{H} = -\frac{\hbar^2}{2} \sum_{k=1}^{n} \frac{\Delta_k}{m_k} + V(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n)$$

which is the sum of the kinetic energy operators of the particles and their potential energy operator. For the case of a system of identical particles we can state that since the permutation of any two particles leaves the expressions of type $|\psi|^2$ and $\int \psi^* \hat{L} \psi \, d\tau$ unchanged, the wave functions $\psi(\mathbf{r}_1, \sigma_1, \mathbf{r}_2, \sigma_2)$ and $\psi(\mathbf{r}_2, \sigma_2, \mathbf{r}_1, \sigma_1)$ can differ only by a factor $e^{i\alpha}$. If we permute these particles a second time, we find that

$$\begin{aligned} \psi(\mathbf{r}_2, \sigma_2, \mathbf{r}_1, \sigma_1) &= \psi(\mathbf{r}_1, \sigma_1, \mathbf{r}_2, \sigma_2)\, e^{i\alpha} \\ &= \psi(\mathbf{r}_2, \sigma_2, \mathbf{r}_1, \sigma_1)\, e^{2i\alpha} \end{aligned}$$

or $e^{i\alpha} = \pm 1$. Thus, a system of identical particles may be described either by symmetric functions ($e^{i\alpha} = +1$ for particles with integral spin) or by antisymmetric functions ($e^{i\alpha} = -1$ for particles with half-integral spin). This is valid for both spatial and spin variables.

If we introduce the spin wave functions ($\alpha(\sigma)$ and $\beta(\sigma)$) for a single electron

$$\alpha(+1) = 1, \quad \alpha(-1) = 0$$

and

$$\beta(+1) = 0, \quad \beta(-1) = 1$$

that is,

$$\alpha(\sigma) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \beta(\sigma) = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

we find that for a system of two electrons whose spins do not interact, the spin wave functions are composed of such products as $\alpha_1\alpha_2$, $\alpha_1\beta_2$, etc., where α_1 belongs to one electron, α_2 to the other, etc.

For the case of an external perturbation $\hat{W}(\mathbf{r}, t)$ causing an unperturbed system (whose Hamiltonian is $\hat{H}_0$) to make a transition from a stationary state ψ_k^0 to another stationary state ψ_l^0, we must use the time-dependent perturbation theory (the theory of quantum transitions).

Denote the eigenfunctions of $\hat{H}_0$ by ψ_n^0, which are the solutions to $\hat{H}_0\psi_n^0 = E_n^0\psi_n^0$. The problem is, given the complete set of ψ_n^0, to find the solution of the Schrödinger equation

$$i\hbar \frac{\partial \Psi}{\partial t} = (\hat{H}_0 + \hat{W})\,\Psi$$

in the form of an expansion

$$\Psi = \sum_n c_n(t)\,\psi_n^0 e^{-iE_n^0 t/\hbar}$$

As an initial condition at $t = 0$ we take the system to be in a particular state ψ_k^0, so that $\Psi(0) = \psi_k^0$, that is, $c_n(0) = \delta_{nk}$. The differential equation for $c_n(t)$ is

$$i\hbar \frac{dc_l}{dt} = \sum_n W_{ln}(t)\, c_n(t)\, e^{i\omega_{ln} t}$$

where $\hat{W}_{ln}(t) = \int \psi_l^{0*}\hat{W}(\mathbf{r}, t)\,\psi_n^0\, d\tau$ is a matrix element of the perturbation energy operator, and $\omega_{ln} = (E_l^0 - E_n^0)/\hbar$.

To solve this equation to a first approximation we must substitute $c_n(0)$ for $c_n(t)$ in the right-hand side and also assume that $\hat{W}(\mathbf{r}, t) = 0$ for $t < 0$ and $t > T$, where T is the time of perturbation. Then

$$c_l(T) = \frac{1}{i\hbar} \int_0^T W_{lk}(t)\, e^{i\omega_{lk} t} dt$$

Since at $t = 0$ the system was in a state ψ_k^0, the probability of transition to a state ψ_l^0 is $|c_l(T)|^2$, i.e.

$$P_{k\to l} = |c_l(T)|^2 = \frac{4\pi^2}{\hbar^2} |W_{lk}(\omega_{lk})|^2 \qquad \text{(III-34)}$$

where $W_{lk}(\omega_{lk}) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} W_{lk}(t)\, e^{i\omega_{lk}t}\, dt$ is the Fourier transform of a matrix element of the perturbation energy operator, corresponding to the Bohr frequency ω_{lk}.

If the transition is caused by the radiation of light (either absorption or emission) whose wavelength is considerably greater than the dimensions of the system (for an atom, $\lambda \gg a_B$), the perturbation energy can be written in the form

$$\hat{W} = -(\mathbf{E}(t)\cdot\mathbf{D})$$

where $\mathbf{E}(t)$ is the electric field vector, and $\mathbf{D} = e\mathbf{r}$ is the dipole moment of the system. And so, in the electric dipole approximation,

$$P_{k\to l} = \frac{4\pi^2}{\hbar^2} |(D_E)_{lk}|^2 |\mathbf{E}(\omega_{lk})|^2 \tag{III-35}$$

From the last relationship we see that under the influence of radiation the system passes from a kth state into an lth state only when $\mathbf{E}(\omega_{lk}) \neq 0$ (that is, if in the radiation there is a frequency equal to the Bohr frequency of this transition), and also when

$$(D_E)_{lk} \neq 0 \tag{III-36}$$

which defines the so-called selection rules.

When a time-independent perturbation $\hat{W}(\mathbf{r})$ is "turned on" over a period of time $0 \leqslant t \leqslant \tau$ (the rest of the time it is "off"), formula (III-34) takes another form:

$$P_{k\to l} = \frac{2\pi}{\hbar} |W_{lk}|^2 \times \tau \times \delta(E_k - E_l)$$

PROBLEMS

Bohr's theory

Using the Bohr-Sommerfeld quantization rule (III-10), solve the following problems.

1. Determine the energy levels of a particle in a one-dimensional potential well with walls of infinite height at $x = 0$ and $x = a$.

2. Quantize the motion of a one-dimensional harmonic oscillator.

3. Find the energy levels for a particle that performs small oscillations in a three-dimensional potential well about its position of equilibrium (this position being in the origin of coordinates). The potential energy $V(x, y, z)$ in the position of equilibrium is zero.

4. Find the energy levels of a particle of mass m that freely rotates in a plane circle of radius r (a rigid plane rotator).

5. Find the energy levels of an electron moving in an elliptical orbit about a nucleus of charge Ze.

6. Determine the energy levels of a hydrogen atom that moves freely in the limits $0 < x < a$, $0 < y < b$, $0 < z < c$.

7. A particle of mass m falls onto a horizontal plane and elastically bounces up. Quantize the particle's motion, determine the admissible heights H_n, and calculate the energy levels of this system.

Operators

8. Find the explicit expressions for the following operators:

(a) $\left(\frac{d}{dx}+x\right)^2$; (b) $\left(\frac{d}{dx}+\frac{1}{x}\right)^3$;

(c) $\left(x\frac{d}{dx}\right)^2$; (d) $\left(\frac{d}{dx}x\right)^2$;

(e) $[i\hbar\nabla+\mathbf{A}(\mathbf{r})]^2$; (f) $(\hat{L}-\hat{M})(\hat{L}+\hat{M})$.

9. Find the commutation relations for the following operators:

(a) x and $\frac{d}{dx}$; (b) $i\hbar\nabla$ and $\mathbf{A}(\mathbf{r})$; (c) $\frac{\partial}{\partial\varphi}$ and $f(r, \theta, \varphi)$.

10. Find the translation operators that map (a) $\psi(x)$ into $\psi(x+a)$; (b) $\psi(\mathbf{r})$ into $\psi(\mathbf{r}+\mathbf{a})$.

Find the operator that rotates space through an angle α.

11. Find the operators that are hermitian conjugate to

(a) $\frac{\partial}{\partial x}$; (b) $\frac{\partial^n}{\partial x^n}$.

12. Check the hermiticity of the operators x, $i\frac{\partial}{\partial y}$, and of the Laplacian operator.

13. Find the operator that is hermitian conjugate to the operator of space translation by vector **a** (see Problem 10b).

14. Find the operator that is hermitian conjugate to $e^{i\alpha\frac{\partial}{\partial\varphi}}$.

15. Find the hermitian conjugate to the product of operators $\hat{A}$ and $\hat{B}$.

16. Show that if $\hat{L}$ and $\hat{M}$ are hermitian, the operators

$$\hat{F}=\frac{1}{2}(\hat{L}\hat{M}+\hat{M}\hat{L}) \quad \text{and} \quad \hat{f}=\frac{i}{2}(\hat{L}\hat{M}-\hat{M}\hat{L})$$

are also hermitian.

17. Prove that the expectation value of the square of an observable is always positive.

18. For operators $\hat{L}$ and $\hat{M}$ satisfying the condition $\hat{L}\hat{M}-\hat{M}\hat{L}=1$, find $\hat{L}\hat{M}^2-\hat{M}^2\hat{L}$.

19. For operators $\hat{L}$ and $\hat{M}$ satisfying the condition $\hat{L}\hat{M}-\hat{M}\hat{L}=1$ (see Problem 18), find $f(\hat{L})\hat{M}-\hat{M}f(\hat{L})$.

20. For any two operators $\hat{A}$ and $\hat{B}$ that do not commute prove the validity of the following relationships (provided that $\hat{A}^{-1}$ exists):

(a) $\hat{A}^{-1}\hat{B}^2\hat{A}=(\hat{A}^{-1}\hat{B}\hat{A})^2$;

(b) $\hat{A}^{-1}\hat{B}^n\hat{A}=(\hat{A}^{-1}\hat{B}\hat{A})^n$ if n is an integer;

(c) $\hat{A}^{-1}f(\hat{B})\hat{A}=f(\hat{A}^{-1}\hat{B}\hat{A})$.

21. Prove that the relationship $e^{\zeta\hat{A}}\hat{B}e^{-\zeta\hat{A}}=\hat{B}+C\zeta$ holds true provided that $\hat{A}\hat{B}-\hat{B}\hat{A}=C$, where C is a scalar and ζ is a parameter.

22. Prove that

$$e^{i\zeta\hat{p}/\hbar}F(\hat{q})\,e^{-i\zeta\hat{p}/\hbar}=F(\hat{q}+\zeta)$$

where $\hat{p}$ and $\hat{q}$ are the momentum and position operators, respectively.

23. Find the eigenfunctions and eigenvalues for the operators $\frac{d}{dx}$ and $i\frac{d}{dx}$.

24. Find the eigenfunctions and eigenvalues for the operator $x + \frac{d}{dx}$.

25. Find the eigenfunctions and eigenvalues for the operator $\frac{d}{d\varphi}$.

26. Find the eigenfunctions and eigenvalues for $\sin \frac{d}{d\varphi}$.

27. Find the eigenfunctions and eigenvalues for $\cos \left(i \frac{d}{d\varphi}\right)$.

28. Find the eigenfunctions and eigenvalues for $e^{ia\frac{d}{d\varphi}}$.

29. Find the eigenfunctions and eigenvalues for $\frac{d^2}{dx^2} + \frac{2}{x}\frac{d}{dx}$.

30. Make the transition from the classical Poisson bracket to the quantum Poisson bracket assuming that their properties are the same. For instance, the relation

$$\{f, g\varphi\} = g\{f, \varphi\} + \{f, g\}\varphi$$

(where f, g, φ are functions corresponding to observables, and $\{\ \}$ is the Poisson bracket) holds true for the operators $\hat{f}$, $\hat{g}$, $\hat{\varphi}$.

31. Find the commutation relation for the annihilation and creation operators $\hat{a}$ and $\hat{a}^*$, where $\hat{a} = \frac{1}{\sqrt{2\hbar\omega}}(\omega\hat{q} + i\hat{p})$ and $\hat{a}^* = \frac{1}{\sqrt{2\hbar\omega}}(\omega\hat{q} - i\hat{p})$.

32. Express the Hamiltonian $\hat{H}$ for the simple harmonic oscillator in terms of the annihilation and creation operators (see Problem 31).

33. Prove the commutation relation for the operator of angular momentum $\hat{\mathbf{L}}$: $[\hat{\mathbf{L}} \times \hat{\mathbf{L}}] = i\hbar\hat{\mathbf{L}}$.

34. Prove that the operator $\hat{\mathbf{L}}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2$ commutes with all three components of $\hat{\mathbf{L}}$ (use the results obtained in Problem 33).

35. Check the following commutation relations for the operator of the dipole moment of a system of N electrons,

$\hat{\mathbf{d}} = -e \sum_{i=1}^{N} \mathbf{r}_i$, and the operator of the system's total angular momentum $\hat{\mathbf{L}} = \sum_{i=1}^{N} \hat{\mathbf{L}}_i$:

$$[\hat{L}_x, \hat{d}_x] = 0, \ [\hat{L}_x, \hat{d}_y] = i\hbar \hat{d}_z, \ [\hat{L}_x, \hat{d}_z] = -i\hbar \hat{d}_y, \text{ etc.,}$$
$$x, y, z \text{ cyclic.}$$

36. Show that

$$[\hat{\mathbf{L}}^2, \hat{\mathbf{d}}] = 2\hbar^2 \hat{\mathbf{d}} + 2i\hbar [\hat{\mathbf{d}} \times \hat{\mathbf{L}}]$$

where $\mathbf{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2$ is the operator of the square of the total angular momentum, and $\hat{\mathbf{d}}$ is as defined in Problem 35.

Solution of the Schrödinger equation. Calculation of expectation values and probability currents

37. Find the general solution of the one-dimensional time-dependent Schrödinger equation for a free particle.

38. At time $t = 0$ the state of a free particle is specified by a wave function

$$\Psi(x, 0) = Ae^{-\frac{x^2}{a^2} + ik_0 x}$$

Find the factor A and the region where the particle is localized. Determine the probability current density $\mathbf{j}$.

39. Find the Fourier transform of the wave function of Problem 38. Determine the width of the wave packet in k-space.

40. A wave packet is represented at the time $t = 0$ by a function

$$\Psi(x, 0) = Ae^{-\frac{x^2}{a^2} + ik_0 x}$$

How will it propagate in time, i.e. what will be the form of the wave function $\Psi(x, t)$? Also determine the probability density $\rho(x, t)$ and the probability current density $\mathbf{j}(x, t)$.

41. Find the expectation values of position and momentum for a particle with a wave function $\psi(x) = Ae^{-x^2/a^2+ik_0x}$.

42. Calculate $\langle \Delta x^2 \rangle$ and $\langle \Delta p^2 \rangle$ for the particle of Problem 41 and check the uncertainty relation for these two quantities.

43. A particle is in a one-dimensional potential well $0 \leqslant x \leqslant a$, for which $V = 0$ inside the well, and $V = \infty$ outside. Solve the time-independent Schrödinger equation for this case.

44. Find the wave function and the allowed energy levels for a particle in a potential field $V(x)$ of the form

$$\begin{aligned} V(x) &= 0 \text{ for } 0 \leqslant x \leqslant a, \quad 0 \leqslant y \leqslant b, 0 \leqslant z \leqslant c \\ &= \infty \text{ for } x < 0, x > a, \quad y < 0, \quad y > b, \ z < 0, \ z > c. \end{aligned}$$

45. Find the energy levels and the wave functions for a particle in a rectangular potential well of finite depth (the one-dimensional case). The field $V(x)$ is given in the form

$$\begin{aligned} V(x) &= 0 && \text{for} \quad x < -a && \text{(in the first region)} \\ &= -V_0 && \text{for} \quad -a \leqslant x \leqslant a && \text{(in the second region)} \\ &= 0 && \text{for} \quad x > a && \text{(in the third region).} \end{aligned}$$

46. Find the energy levels of a three-dimensional harmonic oscillator whose potential energy is

$$V = \frac{k_1 x^2}{2} + \frac{k_2 y^2}{2} + \frac{k_3 z^2}{2}.$$

47. Find the energy levels and the wave functions of a one-dimensional harmonic oscillator that is located in a constant electric field **E**. The electric charge of the oscillator is e.

48. Consider a one-dimensional harmonic oscillator that is on its nth energy level. Find $\langle x^2 \rangle$ and the expectation value of potential energy for such a case.

49. Find the average kinetic energy of a one-dimensional harmonic oscillator whose energy is $\frac{7}{2}\hbar\omega$.

50. Solve the Schrödinger equation for a particle in a potential field $V = V_0 (e^{-2\alpha x} - 2e^{-\alpha x})$.

51. Find the energy levels and the wave functions of a particle in a one-dimensional Coulomb potential well $V(x) = -\frac{e^2}{|x|}$.

52. Solve the Schrödinger equation for a three-dimensional, spherically symmetric harmonic oscillator with a potential energy $V(r) = \frac{m\omega^2}{2} r^2$.

53. Solve the two-dimensional Kepler problem, i.e. find the energy spectrum and the wave functions of a particle in a potential field $V = -\frac{Ze^2}{\rho}$, where $\rho = \sqrt{x^2+y^2}$. (All functions are independent of z.)

54. Solve the Schrödinger equation for a particle in an infinitely deep, spherically symmetric potential well specified by the potential

$$\begin{aligned} V(r) &= 0 \quad \text{for} \quad r < a \\ &= \infty \quad \text{for} \quad r > a. \end{aligned}$$

55. The electron of a hydrogen atom is in its ground state. Determine $\langle r \rangle$, $\langle r^2 \rangle$, and the most probable value r_0 for this case.

56. Normalize the wave functions of the electron in a hydrogen atom corresponding to $n = 2$, where n is the principal quantum number.

57. Consider a particle in a potential field

$$V(r) = -\frac{e^2}{r} + \frac{C}{r^2}$$

Find the energy levels and the corresponding wave functions for the system.

58. Solve the Schrödinger equation for a particle in a potential field $V(r) = Ar^2 + \frac{B}{r^2}$.

59. Consider a particle that in the yz-plane moves freely in a rectangle $0 \leqslant y \leqslant a$, $0 \leqslant z \leqslant b$, whereas the rest of the plane is inaccessible for the particle. Along the x-axis the particle is acted upon by a quasi-elastic force $F = -kx$. Find the energy levels and the corresponding wave functions for the particle, and calculate the normalization factor.

60. Solve the Schrödinger equation for the electron of a hydrogen atom in parabolic coordinates.

61. Solve the Schrödinger equation for a particle with a zero angular momentum ($l = 0$) in a potential field $V = -V_0 e^{-r/a}$.

62. What is the energy spectrum of a particle in a periodically changing potential field? The field is given by the relations

$$V(x) = 0 \quad \text{for} \quad nl \leqslant x \leqslant nl + a \qquad (n = 0, \pm 1, \pm 2, \ldots)$$
$$= V_0 \text{ for } nl - b \leqslant x \leqslant nl$$

The quantity $l = a + b$ is the period of the potential.

63. Examine Problem 62 for the case when $V = 0$ everywhere except at points $x = nl$. In these points $V_0 = \infty$ and $b \to 0$ in such a way that

$$\lim \frac{mV_0 b}{\hbar^2} = \text{constant}$$

(the Kronig-Penney model). Find the dependence of the energy E on the wave vector **k** near the edge of the allowed energy bands.

64. Examine the case of a semi-infinite crystal with a periodically changing potential when $x > 0$, determined as in Problem 63. In the region $x < 0$ the potential $V = W_0$. Restrict yourself to the case when $E < W_0$ (the Tamm levels).

65. Using the uncertainty relations for $\hat{p}$ and $\hat{x}$, estimate the energy of the ground state of a one-dimensional harmonic oscillator.

66. Two particles that interact by an elastic force $F = k(x_1 - x_2)$ can freely move along the x-axis (one-dimensional motion only). Find the energy spectrum and the wave function for this system.

67. Find the energy spectrum and the wave function of the hydrogen atom, taking account of the participation of the nucleus in the relative motion about the centre of mass.

68. Two particles of mass m are able to move along the x-axis only and interact with each other by an elastic force.

In addition, each of them interacts with the origin $x = 0$ by another elastic force with a force constant different from the first. Find the energy levels and the wave functions for such a system.

Theory of representations. Matrices

69. A particle is in an infinitely deep, one-dimensional well (see Problem 43) in a state with an energy $E_2 = \frac{4\pi^2\hbar^2}{2ma^2}$. Determine the momentum distribution for this particle.

70. Find the operator $\hat{x}$ in the momentum representation. Determine its eigenvalues and eigenfunctions.

71. For a particle in a homogeneous potential field find the eigenvalues and eigenfunctions of its Hamiltonian in the momentum representation.

72. Determine the matrix elements of the dipole moment operator and of $\hat{x}^2$ and $\hat{p}$ for a particle in an infinitely deep, one-dimensional well, $-\frac{a}{2} \leqslant x \leqslant \frac{a}{2}$.

73. Find the eigenfunctions for the Hamiltonian of a one-dimensional harmonic oscillator using the momentum representation.

74. Find the energy eigenvalues and the matrix elements of the position and momentum operators in the energy representation by using only the commutation relations for $\hat{p}$ and $\hat{q}$.

75. Use the commutation relations for the components of angular momentum to find the eigenvalues of the square of angular momentum, $\hat{\mathbf{L}}^2$, the z-component of angular momentum, $\hat{L}_z$, and the matrix elements of $\hat{L}_x$ and $\hat{L}_y$ in the $(\hat{\mathbf{L}}^2, \hat{L}_z)$-representation.

76. A wave function $\psi = Ax(a - x)$ specifies the state of a particle in an infinitely deep potential well with walls at $x = 0$ and $x = a$. Find the energy distribution, the expectation value $\langle E \rangle$, and the variance of the energy distribution $\langle \Delta E^2 \rangle$.

77. A plane rotator is in a state with a wave function $\psi = A \sin^2 \varphi$. Determine the probability of finding diffe-

rent values of the z-component of angular momentum, L_z, and the expectation values $\langle L_z \rangle$ and $\langle L_z^2 \rangle$.

78. Find the wave functions in the x- and p-representations for a particle localized at point x_0 and for a particle with a definite momentum p_0.

79. Find the expression for the operator $\frac{1}{r}$ in the momentum representation.

80. Calculate the angular part of the matrix elements of the dipole moment for a particle in a spherically symmetric field.

Time dependence of operators. Potential barriers. Integrals of motion

81. Check whether the following equation for operators holds true:

$$\frac{d}{dt}(\widehat{AB}) = \frac{d\hat{A}}{dt}\hat{B} + \hat{A}\frac{d\hat{B}}{dt}.$$

82. Construct the operators $\frac{d\hat{\mathbf{r}}}{dt}$ and $\frac{d\hat{\mathbf{p}}}{dt}$.

83. Find the formula for the expectation value for current density if the operator $\hat{\mathbf{j}}$ is defined by the classical formula, $\mathbf{j} = \rho\delta(\mathbf{r} - \mathbf{r}_0)\,\mathbf{v}$.

84. Determine under what conditions the square of angular momentum, $\hat{\mathbf{L}}^2$, and the z-component of angular momentum, $\hat{L}_z$, can be integrals of motion.

85. For a particle whose potential energy is $\frac{\alpha}{r}$ construct an operator that is the quantum analog of the following quantity in classical mechanics:

$$\mathbf{K} = [\mathbf{v} \times \mathbf{L}] + \frac{\alpha \mathbf{r}}{r}$$

Show that $\hat{K}_x$, $\hat{K}_y$, and $\hat{K}_z$ are integrals of motion (compare the results with those of Problem 34, Section I).

86. Construct the time derivatives $\frac{d\hat{a}}{dt}$ and $\frac{d\hat{a}^*}{dt}$ for the operators $\hat{a}$ and $\hat{a}^*$ of Problem 31.

87. Find the equations of motion for a system with a Hamiltonian

$$\hat{H}=\frac{(\hat{\mathbf{p}}-e\mathbf{A})^2}{2M}+e\varphi(\mathbf{r},t),\quad \text{where}\quad \mathbf{A}=\mathbf{A}(\mathbf{r},t).$$

88. Find $\frac{d\hat{x}_i}{dt}$ and $\frac{d}{dt}(\hat{p}_i-eA_i)$ if the Hamiltonian of the particle under consideration is of the form

$$\hat{H}=\sum_{k=1}^{3}c\alpha_k(\hat{p}_k-eA_k)+m_0c^2\alpha_4+e\varphi(x_1,x_2,x_3)$$

where $\mathbf{A}=\mathbf{A}(x_1,x_2,x_3,t)$ and α_k, α_4 are matrices that satisfy the following conditions: $\alpha_i\alpha_k+\alpha_k\alpha_i=2\delta_{ik}$.

89. Use the definitions of Problem 88 and its solution to make sure that $\left(\frac{d\hat{x}_i}{dt}\right)_{\text{even}}=\frac{\hat{p}_i}{m_{\text{rel}}}$ if $\hat{\mathbf{A}}_{\text{even}}=\frac{1}{2}(\hat{\varepsilon}\hat{\mathbf{A}}+\hat{\mathbf{A}}\hat{\varepsilon})$, where $\hat{\varepsilon}=\frac{\hat{H}}{|E|}$ (E is an eigenvalue of $\hat{H}$).

90. Show that for a system of N particles the total momentum is an integral of motion, provided there are no external forces acting on the system.

91. A particle moving in the positive direction of the x-axis meets a potential step, the form of the potential being

$$\begin{aligned} V(x)&=0 &&\text{for}\quad x<0\\ &=V_0 &&\text{for}\quad x>0\end{aligned}$$

Determine the wave function for $E>V_0$ and for $E<V_0$, then for both cases calculate the probability current densities for the incident, reflected, and transmitted waves and also the reflectance and the transmittance.

92. Calculate the reflectance and the transmittance when a flux of particles strikes a rectangular potential barrier of width a:

$$\begin{aligned} V(x)&=0 &&\text{for}\quad x<0\\ &=V_0 &&\text{for}\quad 0\leqslant x\leqslant a\\ &=0 &&\text{for}\quad x>a\end{aligned}$$

Both quantities are ratios of the corresponding current densities to the current density of the incident wave.

93. Examine the behaviour of a particle in the following potential field:

$$\begin{aligned} V(x) &= \infty && \text{for} && x < 0 \\ &= 0 && \text{for} && 0 \leqslant x \leqslant a \\ &= V_0 && \text{for} && a \leqslant x \leqslant b \\ &= 0 && \text{for} && b < x \end{aligned}$$

Restrict yourself to the case when $E < V_0$. Study the wave function in the case when its amplitude in the inner region ($0 \leqslant x \leqslant a$) is considerably smaller than in the outer region ($x \geqslant b$).

94. Calculate the transmittance and the electric current density caused by the emission of electrons from a metal due to an electric field **E**. The surface of the metal is in plane $x = 0$.

95. Find the decay constant λ if for alpha-decay the transmittance of the potential barrier D and λ are related as $\lambda = nD$ and if the potential is of the form

$$V = -V_0 \quad \text{for} \quad r < r_0$$

and for $r \geqslant r_0$, the alpha-particle interacts with the nucleus via the Coulomb law. The electric charge of the nucleus is Ze, and $r_0 \ll \frac{2Ze^2}{E}$. The factor $n \propto \frac{v_i}{r_0}$, where v_i is the particle's velocity inside the nucleus and r_0 is the radius of the nucleus, is proportional to the number of collisions with the "wall" of the nucleus per unit time.

A particle in a magnetic field. Spin

96. Determine the energy levels of a free electron in a uniform magnetic field, with the vector of magnetic induction **B** directed along the z-axis.

97. Show that if we substitute the vector potential $\mathbf{A}' = \mathbf{A} + \operatorname{grad} f$ for $\mathbf{A}$ and $\varphi' = \varphi - \frac{\partial f}{\partial t}$ for φ in the time-dependent Schrödinger equation, this leads to a change in the wave function that has no physical consequences.

98. Construct the probability current density for a particle in a magnetic field.

99. Prove that the probability current density of Problem 98 is invariant under the transformations specified in Problem 97.

100. Perform a canonical transformation $\psi = \hat{S}\varphi$, where $\hat{S} = \exp\left[\frac{i}{\hbar}\sum_k e_k \mathbf{r}_k \mathbf{A}(t, 0)\right]$, in the equation

$$i\hbar\frac{\partial\psi}{\partial t} = \left\{\sum_k \frac{[\hat{\mathbf{p}}_k - e_k \mathbf{A}(t, \mathbf{r}_k)]^2}{2m_k} + V\right\}\psi$$

Expand the vector potential in a series

$$\mathbf{A}(t, \mathbf{r}) = \mathbf{A}(t, 0) + (\mathbf{r}\cdot\nabla)\mathbf{A}(t, 0) + \ldots$$

and obtain an equation for φ, assuming that the dimensions of the system under consideration (an atom, for instance) are considerably smaller than the wavelength of the electromagnetic radiation.

101. Prove that the operators $\hat{\sigma}_x$, $\hat{\sigma}_y$, $\hat{\sigma}_z$ defined as follows:

$$\hat{\sigma}_x\alpha = \beta, \quad \hat{\sigma}_y\alpha = i\beta, \quad \hat{\sigma}_z\alpha = \alpha$$
$$\hat{\sigma}_x\beta = \alpha, \quad \hat{\sigma}_y\beta = -i\alpha, \quad \hat{\sigma}_z\beta = -\beta$$

satisfy the same relationships as the Pauli matrices.

102. Construct the operator $\frac{d\hat{\sigma}_x}{dt}$ with the help of the Hamiltonian $\hat{H}$ for a particle with spin 1/2 in a magnetic field with induction **B**.

103. Find the eigenfunctions and eigenvectors for the operators

$$\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

104. What is the projection of the square of spin $\hbar/2$ on a given direction?

105. Check the relationship

$$(\hat{\boldsymbol{\sigma}}\cdot\hat{\mathbf{A}})(\hat{\boldsymbol{\sigma}}\cdot\hat{\mathbf{B}}) = (\hat{\mathbf{A}}\cdot\hat{\mathbf{B}}) + i\hat{\boldsymbol{\sigma}}[\hat{\mathbf{A}}\times\hat{\mathbf{B}}]$$

where $\hat{\boldsymbol{\sigma}}$ is a vector whose components are the Pauli matrices $\hat{\sigma}_x$, $\hat{\sigma}_y$, $\hat{\sigma}_z$.

106. Prove that $\hat{\sigma}_+ = \frac{1}{2}(\hat{\sigma}_x + i\hat{\sigma}_y)$ and $\hat{\sigma}_- = \frac{1}{2}(\hat{\sigma}_x - i\hat{\sigma}_y)$ are conjugate operators. Find the commutation relations for them, and also the commutation relations between $\hat{\sigma}_\pm$ and each of the Pauli matrices $\hat{\sigma}_x$, $\hat{\sigma}_y$, $\hat{\sigma}_z$. Determine $\hat{\sigma}^2_\pm$.

107. Given the operators $\hat{\sigma}_x$, $\hat{\sigma}_y$, $\hat{\sigma}_z$ (the Pauli matrices), prove the following relationships:

(a) $\sin(\hat{\sigma}_x \varphi) = \hat{\sigma}_x \sin\varphi$;

(b) $\cos(\hat{\sigma}_z \varphi) = \cos\varphi$;

(c) $e^{i\hat{\sigma}_y \varphi} = \cos\varphi + i\hat{\sigma}_y \sin\varphi$;

(d) $e^{i\hat{\sigma}_z \varphi}\hat{\sigma}_y e^{-i\hat{\sigma}_z \varphi} = e^{2i\hat{\sigma}_z \varphi}\hat{\sigma}_y$.

108. Prove that for the operators $\hat{\sigma}_+$ and $\hat{\sigma}_-$ defined in Problem 106 the following relationship holds true:

$$(\hat{\sigma}_+\hat{\sigma}_-)^n = \hat{\sigma}_+\hat{\sigma}_-$$

where n is an integer.

109. Assume that the transformation to a new representation is carried out by an operator $\hat{S} = e^{-i\frac{\hat{\sigma}_z}{2}\varphi}$, so that $\psi' = \hat{S}\psi$. Find the operators $\hat{A}'$ and $\hat{B}'$ if in the old representation

$$\hat{A} = \hat{\sigma}_x \sin\varphi + \hat{\sigma}_y \cos\varphi \quad \text{and} \quad \hat{B} = \hat{\sigma}_x \cos\varphi - \hat{\sigma}_y \sin\varphi.$$

110. What are the eigenfunctions of $\hat{S}_z$ and $\hat{\mathbf{S}}^2$ for a system of two weakly interacting particles n and p of spin $\hbar/2$ if $\hat{\mathbf{S}} = \hat{\boldsymbol{\sigma}}_n + \hat{\boldsymbol{\sigma}}_p$? Determine the corresponding eigenvalues of $\hat{S}_z$ and $\hat{\mathbf{S}}^2$.

111. Calculate the scalar product of the spin vector operators of two particles in the triplet and singlet states. The spin of both particles is $\hbar/2$.

112. Show that the operator $(\hat{\boldsymbol{\sigma}}_n \cdot \hat{\boldsymbol{\sigma}}_p)^k$ can be linearly expressed by $(\hat{\boldsymbol{\sigma}}_n \cdot \hat{\boldsymbol{\sigma}}_p)$.

113. Determine the energy levels and the wave functions of a particle with spin $S = 1$ (in units of $\hbar$) if the

Hamiltonian

$$\hat{H} = A\hat{S}_x^2 + B\hat{S}_y^2 + C\hat{S}_z^2$$

where A, B, C are constants.

114. Let an electron pass in y direction through a homogeneous magnetic field with induction $\mathbf{B}$ parallel to the z-axis. The spin of the electron points in the positive direction. When passing the point $y = 0$ at the time $t = 0$, the electron enters an additional homogeneous field with induction $\mathbf{B}'$ parallel to the x-axis. It leaves this auxiliary field at $y = l$ and $t = t_0$. What is the probability of a spin flip during this time interval?

Approximate methods of solving quantum mechanical problems

115. Approximate the energy levels and wave functions of the anharmonic oscillator

$$V(x) = \frac{1}{2} m\omega^2 x^2 + \varepsilon_1 x^3 + \varepsilon_2 x^4$$

by the time-independent perturbation theory, in the first and second orders of the approximation.

116. A spinless particle is exposed to a spherically symmetric field, and its energy levels are E_{nl} (for the unperturbed system). Find the energy levels and the corresponding wave functions in the first approximation of the time-independent perturbation theory if the particle is acted upon by a magnetic field parallel to the z-axis.

117. A rigid plane rotator is placed in a weak electric field $\mathbf{E}$ directed along the x-axis. The charge of the particle is e, and it is at a distance a from the centre. Calculate the energy corrections in the first and second approximations.

118. Find the corrections to the energy and to the wave function of an electron in a periodic potential $V(\mathbf{r})$, this being the perturbation. The lattice constant is $\mathbf{a}$. Consider the case when two energy levels are equal: $E_{\mathbf{k}} = E_{\mathbf{k}+2\pi\mathbf{g}}$, where $\mathbf{g}$ is the vector of the reciprocal lattice, $(\mathbf{ga}) = n$.

119. A hydrogen atom is placed in a homogeneous electric field $\mathbf{E}$ directed along the z-axis. Find the splitting of the

energy level that corresponds to $n = 2$ (n is the principal quantum number).

120. Show that there is no linear Stark effect in the atoms of group I of the Periodic Table since the energy levels of these atoms, E_{nl}, are determined only by n and l. (The Stark effect is the splitting of energy levels in an external homogeneous electric field.)

121. Analyze the Stark effect in a hydrogen atom (calculate the splitting of the nth energy level in an electric field $\mathbf{E}$ directed along the z-axis) by using the solution of the unperturbed problem in parabolic coordinates.

122. Consider a hydrogenlike atom (with the charge of the nucleus Ze), and assume that the nucleus is a uniformly charged sphere of radius r_0. Calculate the energy shifts for the $n = 1$, $l = 0$ state, and for the $n = 2$ states, in the first order of approximation of the time-independent perturbation theory.

123. Suppose a system whose stationary states are Ψ_0 and Ψ_1 is in the state Ψ_0. At the time $t = 0$ a perturbation W not depending on time is switched on. Find the time dependence of $\Psi(t)$ for the perturbed system.

124. Show that if a perturbation $W(r)$ not depending on time is switched on over a period of time $0 \leqslant t \leqslant \tau$ and the rest of the time it is off, the probability of transition from state n to state k is

$$\frac{2\pi}{\hbar} |W_{nk}|^2 \tau\delta(E_n - E_k).$$

125. Calculate the probability of the ionization of a hydrogenlike atom by a plane monochromatic wave. The vector potential of the wave $\mathbf{A}$ has the following components:

$$A_x = A\cos(\omega t - \mathbf{kr}), \quad A_y = 0, \quad A_z = 0$$

Before the perturbation (the wave) is switched on, the electron is near the nucleus Ze and is in the $n = 1$, $l = 0$ state. In the final state the electron can be considered free.

126. Two identical particles are located in an external potential field $V(r)$, and the interaction between them is $\hat{H}_{12}$. Assume that the solution of the Schrödinger equation is known for a single particle, and find the solution for the two particles.

127. Use the solution of Problem 126 to examine the time dependence of the probability that the two particles will remain in their original state if at the initial time ($t = 0$) one was in the rth state and the other in the sth state of the single-particle Hamiltonian. Determine the time that is needed for them to change states.

128. Using the Ritz variational method, find the ground state of a three-dimensional oscillator. Choose the trial function in the form

$$\varphi = A\,(1 + \alpha r)\,e^{-\alpha r}.$$

129. Consider the central-force model of the deuteron using the Ritz method. The neutron-proton interaction is idealized by the central-force potential $V(r) = -Ae^{-r/a}$. Choose the trial function in the form

$$\varphi = Be^{-\alpha r/2a}.$$

130. Using the Ritz method with two trial functions, $\varphi_1 = A\,(1 + \alpha r)e^{-\alpha r}$ and $\varphi_2 = Be^{-\alpha r^2/2}$, find the energy of the ground state of the electron in a hydrogen atom and compare the results.

131. Calculate the ionization potential for a helium atom using the method of the perturbation theory and varying the screening constant s. Assume that in the unperturbed system each electron interacts with the nucleus via a potential $-\frac{(2-s)\,e^2}{r}$, and include the compensating member $\frac{se^2}{r_1} + \frac{se^2}{r_2}$ in the perturbation.

132. Consider the elastic scattering of particles by a centre, assuming that the particles interact with the centre via a potential $V(r)$, which is considered a small perturbation. Find the differential scattering cross section, defining it as the ratio of the radial flux through an area element dS far from the scattering centre to the incident flux:

$$d\sigma = \frac{|\mathbf{j}_{\text{scat}}\, dS|}{|\mathbf{j}_{\text{in}}|}.$$

133. Use the result obtained in Problem 132 to find the differential cross section for Coulomb scattering (the Rutherford formula).

134. Determine the differential cross section for the elastic scattering of positively charged particles (each with charge e_1) by an atom. Regard the atom as a fixed centre with charge Ze surrounded by a charged uniform sphere of $-e\rho(r)$ density. Calculate $d\sigma$ for $\rho=\rho_0 e^{-r/a}$.

135. Calculate the differential and total cross sections for the scattering of particles by a field $V=\frac{A}{r}e^{-\varkappa r}$ (the Yukawa potential).

SECTION IV

Statistical Physics and Thermodynamics

Two methods are used to describe the state of systems that consist of a great number of particles, the statistical and the thermodynamic. The first method enables us not only to obtain the general relationships of the second method but to calculate the concrete values of the thermodynamic observables for the given system.

Postulates of classical statistical mechanics. Let us assume that a system consists of N particles contained in volume V. The motion of the particles follows the equations (laws) of classical mechanics. The state of this system is fully determined by defining generalized coordinates q_1, q_2, . . ., q_N and momenta p_1, p_2, . . ., p_N (a point in phase space of $2N$ dimensions), and the dynamics by the Hamiltonian $H = H(p_i, q_i)$, which makes it possible to find p_i, q_i at any instant of time by means of the equations

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}, \quad \dot{q}_i = \frac{\partial H}{\partial p_i} \quad (i = 1, 2, \ldots, N) \qquad \text{(IV-1)}$$

It is impossible to determine the state of a system with a large number of particles (N) at any instant of time, and there is no need to. A study of these many-particle systems usually focusses on some small number of macroscopic variables. For example, suppose we require that the number of particles in the system is N, the volume is V, and that the possible values of energy lie in the interval $[E, E + \Delta E]$. Obviously, many states of the system satisfy these requirements. Hence, to calculate for a given system the average of a macroscopic variable of the type

$$\widetilde{F}^t = \lim_{T \to \infty} \frac{1}{T} \int_0^T F[p_i(t), q_i(t)]\, dt \qquad \text{(IV-2)}$$

we use the following method.

We introduce an infinite number of copies of the given system [one Hamiltonian $H(p_i, q_i)$ but different initial conditions p_i^0, q_i^0] at some time t. Such an "ensemble" of systems will be distributed in phase space with a certain density $\rho(p_i, q_i, t)$. If the system is isolated from the ambient medium, its energy will be a constant value, i.e.

$$H(p_i, q_i) = E \tag{IV-3}$$

and the phase points of the ensemble will be distributed over the hypersurface of constant energy (IV-3).

The probability density is determined in this case by the following relationship known as the microcanonical distribution:

$$\rho(H) = \frac{\delta[H(p_i, q_i) - E]}{\Omega(E)} \tag{IV-4}$$

where $\delta(x)$ is the Dirac delta function (see Appendix 4), and

$$\Omega(E) = \frac{\partial \Gamma(E)}{\partial E}, \quad \Gamma(E) = \int \cdots \int_{H(p_i, q_i) \leqslant E} d^N p d^N q \tag{IV-5}$$

The ensemble average for any physical observable will now be

$$\overline{F^s} = \int \underset{-\infty}{\overset{\infty}{\cdots}} \int \rho(H) F(p_i, q_i) d^N p d^N q \tag{IV-6}$$

Real physical systems are in contact with the ambient medium. For example,

(a) the mechanical interaction with a source of work that affects the system. Such a state of the system can be described by a Hamiltonian $H(p_i, q_i, a_s)$, where a_s are the generalized coordinates of external bodies viewed as additional variables of the system. Then $A_s = -\frac{\partial H}{\partial a_s}$ will stand for the generalized force which the system exerts on the ambient medium;

(b) the thermal contact between the system described by a Hamiltonian $H(p_i, q_i, a_s)$ and a heat bath (another system) with a Hamiltonian $H_0(p_i', q_i')$. The total Hamiltonian is

$$H(p_i, q_i, p_i', q_i', a_s) = H(p_i, q_i, a_s) + H_0(p_i', q_i') + H'(p_i, q_i, p_i' \; q_i')$$

These systems are in equilibrium if the energy H' is small and allows the system to exchange energy with the heat bath fast enough for any state of the total system to be realized after a long period of time with equal probability;

(c) the material contact between the system and a heat bath that consists of an exchange of particles. In a state of equilibrium the energy H' must satisfy the previous requirements.

In the case of (b) the probability density for the ensemble has the form (the canonical distribution)

$$\rho(H) = Ce^{-H(p_i,\, q_i,\, a_s)/\Theta} \qquad \text{(IV-7)}$$

where C is the normalization constant, and

$$\frac{1}{\Theta} \equiv \frac{\partial \ln \Omega(E)}{\partial E} \qquad \text{(IV-8)}$$

Here Θ is a constant common to both systems in equilibrium.

Denoting $C \equiv e^{F(\Theta,\, a_s)/\Theta}$, we get

$$F(\Theta, a_s) = -\Theta \ln \int \overset{\infty}{\underset{-\infty}{\cdots}} \int e^{-H(p_i,\, q_i,\, a_s)/\Theta} d^N p\, d^N q \qquad \text{(IV-9)}$$

The quantity

$$Z \equiv \int \overset{\infty}{\underset{-\infty}{\cdots}} \int e^{-H(p_i,\, q_i,\, a_s)/\Theta} d^N p\, d^N q \qquad \text{(IV-10)}$$

has come to be known as the classical partition function. It is the main quantity when the physical properties of a system are evaluated.

From formula (7) we can show that

$$E \equiv \overline{H} = F - \Theta \left(\frac{\partial F}{\partial \Theta}\right)_{a_s}, \quad \overline{A}_s = -\left(\frac{\partial F}{\partial a_s}\right)_{\Theta} \qquad \text{(IV-11)}$$

and the entropy is defined as

$$S = -k\left(\frac{\partial F}{\partial \Theta}\right)_{a_s}$$

where k is Boltzmann's constant. Since the temperature of two systems in equilibrium in contact with each other is the same in both systems, this suggests that Θ is a function of temperature. We introduce the absolute temperature of a system, T, assuming that $\Theta = kT$. The function F, defined

by formula (IV-9), can be identified with the Helmholtz free energy.

So, knowing the Hamiltonian of the system H, we can find Z according to (IV-10), and then

$$F(T, a_s) = -kT \ln Z \tag{IV-12}$$

All other quantities we find by simple differentiation:

$$E = F - T\left(\frac{\partial F}{\partial T}\right)_{a_s}$$

$$\overline{A_s} = -\left(\frac{\partial F}{\partial a_s}\right)_T, \quad \text{for instance,} \quad p = -\left(\frac{\partial F}{\partial V}\right)_T, \tag{IV-13}$$

$$S = -\left(\frac{\partial F}{\partial T}\right)_{a_s}, \quad \text{etc.}$$

Thermodynamic description of a state of a system. From the relationships just cited we can obtain the basic laws of thermodynamics.

(1) *The first law.* The amount of heat obtained by a system (decrease in the mean energy of the heat bath) goes to change the internal energy of the system and to perform work exerted by the system on external bodies:

$$\begin{gathered} dQ = dE + \sum_s \overline{A_s}\, da_s \\ dQ = -d\overline{H}_0(p_i', q_i') \end{gathered} \tag{IV-14}$$

(2) *The second law.*

$$dS \geqslant \frac{dQ}{T}, \quad T\, dS = dE + \sum_s \overline{A_s}\, da_s \tag{IV-15}$$

The equality corresponds to the equilibrium state of the system.

(3) *The third law* (*Nernst's heat theorem*). As $T \to 0$, the entropy of a system, $S(T, a_s)$, ceases to depend on a_s, i.e. is a permanent quantity for all substances.

The method of thermodynamics rests on these three laws, using the fact that S and E are state functions of the system for which

$$\oint dE = 0, \quad \oint dS = 0 \tag{IV-16}$$

i.e. dE and dS are total, or exact, differentials.

The differential $dZ = Xdx + Ydy$ is an exact differential if

$$\left(\frac{\partial X}{\partial y}\right)_x = \left(\frac{\partial Y}{\partial x}\right)_y$$

Here are the relationships for other functions of state (thermodynamic functions).

(1) The Helmholtz free energy $F = E - TS$:

$$dF = -S\,dT - \sum_s A_s\,da_s$$

$$\left(\frac{\partial S}{\partial a_s}\right)_T = \left(\frac{\partial A_s}{\partial T}\right)_{a_s} \qquad \text{(IV-17)}$$

(2) Enthalpy $H = E + \sum_s A_s a_s$:

$$dH = T\,dS + \sum_s a_s\,dA_s$$

$$\left(\frac{\partial T}{\partial A_s}\right)_{a_s} = \left(\frac{\partial a_s}{\partial S}\right)_{A_s} \qquad \text{(IV-18)}$$

(3) The Gibbs free energy $\Phi = F + \sum_s A_s a_s$:

$$d\Phi = -S\,dT + \sum_s a_s\,dA_s$$

$$-\left(\frac{\partial S}{\partial A_s}\right)_T = \left(\frac{\partial a_s}{\partial T}\right)_{A_s} \qquad \text{(IV-19)}$$

If in addition we use the equation of state $A_s = A_s(T, a_s)$, we get a number of relationships that can be checked experimentally. Thermodynamics does not find the numerical values of the quantities entering into these relationships. [The line above A_s is omitted in formulas (IV-17) to (IV-19).]

Grand canonical distribution. If there are m kinds of particles in a system and the type of contact is (c), we get

$$\rho = Ce^{\left[-H(p_i,\, q_i,\, a_s) + \sum_{j=1}^{m} \mu_j N_j\right]/kT} \qquad \text{(IV-20)}$$

where C is the normalization constant,

$$\frac{1}{kT} \equiv \frac{\partial \ln \Omega(E, \overline{N}_1, \ldots, \overline{N}_m)}{\partial E} \qquad \text{(IV-21)}$$

and

$$\mu_j \equiv kT \frac{\partial \ln \Omega (E, \overline{N}_1, \ldots, \overline{N}_m)}{\partial \overline{N}_j}$$

is the chemical potential of the jth type of particles.

Bearing in mind a correct calculation of the number of states (the Gibbs paradox), we find that

$$Z = \sum_{N_1=0}^{\infty} \cdots \sum_{N_m=0}^{\infty} \frac{e^{\sum_{j=1}^{m} \mu_j N_j / kT}}{N_1!\, N_2! \ldots N_m!} Z_0$$

$$Z_0 = \int \overset{2N}{\cdots} \int e^{-\frac{H(p_i, q_i, a_s)}{kT}} d\Gamma (N_j) \qquad \text{(IV-22)}$$

Defining $C \equiv e^{-\frac{\Xi(T, a_s, \mu_j)}{kT}}$, we get an expression for the thermodynamic function Ξ:

$$\Xi = -kT \ln Z \qquad \text{(IV-23)}$$

and

$$S = -\left(\frac{\partial \Xi}{\partial T}\right)_{\mu_j, a_s}, \quad \overline{A}_s = -\left(\frac{\partial \Xi}{\partial a_s}\right)_{T, \mu_j} \qquad \text{(IV-24)}$$

$$\overline{N}_j = -\left(\frac{\partial \Xi}{\partial \mu_j}\right)_{T, a_s}$$

Hence,

$$d\Xi = -S\, dT - \sum_s \overline{A}_s\, da_s - \sum_{j=1}^{m} \overline{N}_j\, d\mu_j \qquad \text{(IV-25)}$$

$$dS = \frac{1}{T} \left(dE + \sum_s \overline{A}_s\, da_s - \sum_{j=1}^{m} \mu_j\, d\overline{N}_j \right) \qquad \text{(IV-26)}$$

$$dF = -S\, dT - \sum_s \overline{A}_s\, da_s + \sum_{j=1}^{m} \mu_j\, d\overline{N}_j \qquad \text{(IV-27)}$$

$$d\Phi = -S\, dT + \sum_s a_s\, d\overline{A}_s + \sum_{j=1}^{m} \mu_j\, d\overline{N}_j \qquad \text{(IV-28)}$$

$$0 = -S\, dT + \sum_s a_s\, d\overline{A}_s - \sum_{j=1}^{m} \overline{N}_j\, d\mu_j \qquad \text{(IV-29)}$$

We see from the last relationship that μ_j, $\overline{A_s}$, and T are not independent.

The condition for the equilibrium of two systems has the form

$$p_1 = p_2 \quad \text{(mechanical equilibrium)} \qquad \text{(IV-30a)}$$

$$T_1 = T_2 \quad \text{(thermal equilibrium)} \qquad \text{(IV-30b)}$$

$$\frac{\mu_1}{T_1} = \frac{\mu_2}{T_2} \quad \text{(equilibrium with respect to the exchange of particles)} \qquad \text{(IV-30c)}$$

Quantum statistics. The stationary state of a quantum mechanical system of N particles is described by the wave function $\Psi_k(q) = \Psi(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N)$, which is determined from the time-independent Schrödinger equation

$$\hat{H}\Psi_k(q) = E_k \Psi_k(q) \qquad \text{(IV-31)}$$

where

$$\hat{H} = -\frac{\hbar^2}{2} \sum_k \frac{1}{m_k} \Delta_k + U(\mathbf{r}_1, \ldots, \mathbf{r}_N)$$

The value of the physical observable f in a state k of a system (in quantum mechanics an operator $\hat{f}$ corresponds to this observable) is found by the rule

$$\langle f \rangle = \int \Psi_k^* \hat{f} \Psi_k \, dq \qquad \text{(IV-32)}$$

But in statistical physics each state E_k is given a definite probability of realization depending on the macroscopic conditions of the system. It follows from this that the average (mean) value in quantum statistics is calculated thus

$$\langle \bar{f} \rangle \equiv \bar{f} = \sum_k w_k \langle f \rangle_k = \sum_k w_k \int \psi_k^* \hat{f} \psi_k \, dq$$

$$= \int_q \int_{q'} \delta(q - q') \hat{f}(q) \rho(q, q') \, dq \, dq' \qquad \text{(IV-33)}$$

where

$$\rho(q, q') = \sum_k w_k \psi_k^*(q') \psi_k(q) \qquad \text{(IV-34)}$$

is the density matrix in the coordinate representation.

By analogy with classical statistical physics w_k is chosen in the form

$$w_k = e^{\frac{F(a_s, T) - E_k}{kT}} \tag{IV-35}$$

Hence,

$$F = -kT \ln Z \tag{IV-36}$$

where the partition function

$$Z = \sum_k e^{-\frac{E_k}{kT}} = \sum_j e^{-\frac{E_j}{kT}} \Omega(E_j) \tag{IV-37}$$

and $\Omega(E_j)$ is the degeneracy multiplicity of the system's jth energy level. Summing up over j means summing up over the discrete energy levels and integrating over the continuous spectrum of the system. All subsequent relationships are fully equivalent to the relationships of classical statistical mechanics.

When there is material contact,

$$Z = \sum_{N_1=0}^{\infty} \ldots \sum_{N_m=0} \sum_n \exp\left[-\frac{1}{kT}\left(E_{N,n} - \sum_{j=1}^{m} N_j \mu_j\right)\right] \tag{IV-38}$$

where $E_{N,n}$ is the energy of the nth quantum state. The thermodynamic function Ξ is determined as in (IV-23) according to the formula

$$\Xi = -kT \ln Z \tag{IV-39}$$

If the particles are noninteracting, relationship (IV-38) is greatly simplified because the state of this system is given by the occupation numbers n_i owing to the indistinguishability of the particles. Then

$$E_n = \sum_i \varepsilon_i n_i, \quad N = \sum_i n_i \tag{IV-40}$$

where ε_i is the energy of the ith single-particle state, and the partition function is

$$Z = \sum_{n_i} \left(e^{-\frac{\varepsilon_i - \mu}{kT}}\right)^{n_i} \tag{IV-41}$$

where the summing up is done over the number of particles that are in the single-particle state with ε_i.

The general principles of quantum mechanics impose strict rules on the occupation numbers n_i. Only two cases are possible:

(1) $n_i = 0$ or 1 (particles with a half-integral spin);
(2) $n_i = 0, 1, 2, \ldots$ (particles with integral spin).

Now, using (IV-41), we get the following relationships for the average number of particles in a state with energy ε:

$$\bar{n} = \frac{1}{e^{\frac{\varepsilon-\mu}{kT}}+1} \quad \text{(Fermi-Dirac statistics)} \tag{IV-42}$$

$$\bar{n} = \frac{1}{e^{\frac{\varepsilon-\mu}{kT}}-1} \quad \text{(Bose-Einstein statistics)} \tag{IV-43}$$

The energy and the number of particles are determined by standard formulas:

$$E = \sum_i \varepsilon_i \bar{n}_i \tag{IV-44}$$

$$N = \sum_i \bar{n}_i \quad \text{(the condition for finding } \mu)$$

The limit of application of quantum distributions for a perfect gas is

$$\frac{N}{V} \gg \left(\frac{2\pi mkT}{h^2}\right)^{3/2} \tag{IV-45}$$

Fluctuations and kinetic theory. Statistical mechanics makes it possible to calculate the fluctuations of physical quantities because the probability of fluctuation of a macroscopic quantity can be expressed in terms of T and a_s. The probability of fluctuation of a physical quantity x (the equilibrium value of which is x_0) is

$$\rho(x) = Ce^{-\frac{W_{\min}(x_0,\, x)}{kT_0}} \tag{IV-46}$$

where $W_{\min}(x_0, x)$ is the minimal amount of work needed to shift the system from x_0 to x, and T_0 is the equilibrium value of the temperature in the system.

When the deviations from the state of equilibrium are small, for the probability of fluctuations of thermodynamic

quantities we get

$$\rho = Ce^{\frac{\Delta p \Delta V - \Delta T \Delta s}{2kT_0}} \tag{IV-47}$$

With this formula we can find the fluctuations of any thermodynamic observable if we choose the independent variables in the proper way.

In studying various kinetic phenomena we must know the number of particles that at time t have a radius vector lying in the interval $[\mathbf{r}, \mathbf{r} + d\mathbf{r}]$ and a velocity in the interval $[\mathbf{v}, \mathbf{v} + d\mathbf{v}]$, i.e. know

$$f(\mathbf{r}, \mathbf{v}, t)\, d\mathbf{r}\, d\mathbf{v}$$

For the distribution function f we can obtain an integro-differential equation of the type

$$\left[\frac{\partial}{\partial t} + \mathbf{v}_1 \frac{\partial}{\partial \mathbf{r}} + \frac{\mathbf{F}}{m} \frac{\partial}{\partial \mathbf{v}_1}\right] f(\mathbf{r}, \mathbf{v}_1, t)$$
$$= \int d\Omega \int d\mathbf{v}_2 \sigma(\Omega) \left| \mathbf{v}_1 - \mathbf{v}_2 \right|$$
$$\times [f(\mathbf{r}, \mathbf{v}_2', t) f(\mathbf{r}, \mathbf{v}_1', t) - f(\mathbf{r}, \mathbf{v}_2, t) f(\mathbf{r}, \mathbf{v}_1, t)] \tag{IV-48}$$

Here $\mathbf{v}_1$ and $\mathbf{v}_2$ are the velocities of the first and second particles before collision, $\mathbf{v}_1'$ and $\mathbf{v}_2'$ the same quantities after collision, $\sigma(\Omega)$ is the effective cross section of the collision, and $d\Omega$ is the solid angle differential.

To get (IV-48) we must (a) consider none but binary collisions, (b) neglect the action of the walls of the vessel containing the particles, (c) neglect the influence of external forces on the effective cross section, and (d) consider the velocities being independent from the position of the particles in the vessel.

Function f defined in (IV-48) makes it possible to calculate a variety of kinetic coefficients: the tensor of electrical conductivity, the coefficients of thermal conductivity, viscosity, diffusion, etc.

PROBLEMS

1. A certain system can with equal probability be in any of its N states. What is the probability of the system being in one of its states?

2. A simple pendulum performs harmonic oscillations according to the law

$$\varphi = \varphi_0 \cos \frac{2\pi}{T} t \quad \left(T = 2\pi \sqrt{\frac{l}{g}}\right)$$

Determine the probability that in a random measurement of the angle of deviation its value lies in the interval $[\varphi, \varphi + d\varphi]$.

3. The probability that for a certain system the values of x and y lie in the intervals $[x, x + dx]$ and $[y, y + dy]$ is

$$dW(x, y) = Ce^{-\alpha(x^2+y^2)}\, dx\, dy \quad (\alpha > 0)$$

Assuming that the range of values of x and y is $[-\infty, \infty]$, find the normalization constant C.

4. Determine for the previous case the probability that the value of x lies in $[x, x + dx]$.

5. During thermionic emission electrons leave the surface of a metal or a semiconductor. Assuming that (1) emission of electrons are statistically independent events and (2) the probability of emission of an electron in a small time interval dt is λdt (λ being a constant), determine the probability of emission of n electrons in a time interval t.

6. For the previous case determine

$$\overline{\Delta n^2} = \overline{(n - \bar{n})^2}$$

if on the average n_0 electrons are emitted every second.

7. A perfect gas consisting of N molecules is contained in a vessel with a volume V. What is the probability that there will be n molecules at any given time in a volume V_0 (V_0 is much less than V)? Examine the extreme cases: (a) n is much less than N, and (b) n is much greater than unity and $\Delta n = n - \bar{n} \ll \bar{n}$.

8. Prove that for a random variable x the probability of an event in which x becomes greater than a certain value a satisfies the Chebyshev inequality

$$w(x > a) \leqslant \frac{\overline{x^2}}{a^2}.$$

9. A particle that at the initial time is at the origin of coordinates jumps the next instant by one unit to the right

or left with the probability of 1/2. Determine the probability $P_t(l)$ that after t steps the particle will be at point l of the one-dimensional grating (Fig. 44).

10. Determine the probability $P_t(l)$ of a similar random walk of a particle across a two-dimensional (square) and a three-dimensional (cubic) grating if the particle can move

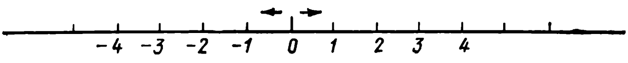

Fig. 44

to any adjacent point (4 points for the square and 6 for the cubic) with the probabilities of 1/4 and 1/6, respectively (Fig. 45).

11. The following question (first raised by G. Polya) arises in considering Problem 9: Can a particle always return

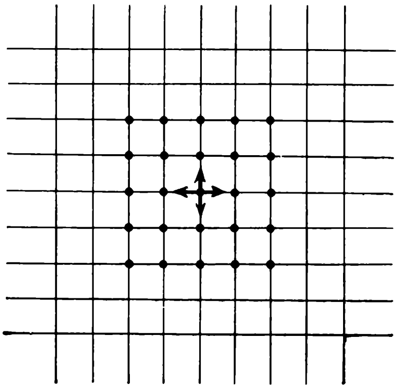

Fig. 45

to its initial point if it moves at random to adjacent points of the grating? If it cannot, what is the probability of its not returning to the initial point for the cases examined in Problems 9 and 10?

12. In space p, q draw the phase trajectory of a particle that moves with constant velocity in a direction perpendicular to the mirror-reflecting walls of a box. The size of the box in the direction of motion is $2a$.

13. Determine the phase trajectory of a body of mass m that moves with an initial velocity v_0 in a constant gravitational field from point z_0 in a vertically upward direction. Draw the trajectory.

14. Determine and draw the phase trajectory for a particle of mass m with an electric charge $-e$ that moves under the influence of a Coulomb force of attraction toward a fixed charge $+e_1$. The initial distance between the charges is r_0, and the initial velocity of the negative charge is $v_0 = 0$.

15. Determine and draw the phase trajectory for a linear harmonic oscillator described by the equation

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = 0 \quad \left(\omega_0 = \sqrt{\frac{k}{m}}\right)$$

where $\gamma \ll \omega_0$. Find the change in the phase volume with the passage of time.

16. Determine and draw the phase trajectory for a compound (physical) pendulum of mass m whose moment of inertia is I and whose equivalent length is L. Consider three cases:

(1) $H_0 > 2mgL$;
(2) $H_0 = 2mgL$;
(3) $H_0 < 2mgL$ (H_0 is the initial energy of the pendulum).

17. Verify whether Liouville's theorem holds for the following cases:

(1) an elastic collision of two spheres (central impact);
(2) the motion of three particles in a constant gravitational field, the initial states of the particles being determined by the phase points

$$A\ (p_0,\ z_0),\quad B\ (p_0,\ z_0 + a),\quad C\ (p_0 + b,\ z_0).$$

18. Determine the normalization divisor of the microcanonical distribution for the following systems: (1) N molecules of perfect monatomic gas; (2) N independent linear oscillators.

19. Derive the canonical distribution using the examples in Problem 18 as the model of a heat bath.

20. We can represent the Hamiltonian of a perfect gas in the form

$$H = \sum_i H_i$$

where H_i is the Hamiltonian of an individual molecule.

Express the classical partition function of the gas in terms of the partition function of an individual molecule. Determine the average energy E, the entropy S and the pressure p.

21. Determine E, S, p, and C_V (C_V is the molar heat capacity at constant volume) for the following systems consisting of N noninteracting particles contained in volume V:

(1) a monatomic gas;

(2) a diatomic gas with hindered oscillations of the atoms (a rigid rotator);

(3) a diatomic perfect gas allowing for the vibrations of atoms in the molecule (consider the case of low temperatures).

22. Determine the energy and pressure for a perfect gas that consists of N particles and is contained in a vessel with a volume V when the energy of an individual particle depends on momentum $\mathbf{p}$ in the following way:

(a) $H = ap^l$, $a > 0$ and $l > 0$;

(b) $H = c\,(m^2c^2 + p^2)^{1/2}$, where c is the velocity of light.

23. The classical partition function of a system is expressed as

$$Z(\beta) = \frac{A}{\beta^N}, \quad \text{where} \quad \beta \equiv \frac{1}{kT}$$

Determine $\Omega(E)$.

24. A system as a whole revolves with an angular velocity $\mathbf{\Omega}$. Find the canonical distribution in the revolving system of coordinates.

25. A cylinder of height h and base radius R is filled with a perfect gas. The cylinder rotates with an angular velocity Ω about an axis perpendicular to the base and passing through its centre. Determine the pressure of the gas on the surface of the cylinder if the number of particles in the gas is N and the mass of an individual particle is m.

26. Derive the virial theorem for a system of interacting particles contained in a volume V. The potential energy of interaction is a homogeneous function of coordinates of degree n.

27. Determine the average value $\overline{H^n}$ ($n > 0$) for a monatomic perfect gas consisting of N particles. Use the result

to find the mean square fluctuation in energy $\overline{\Delta H^2} = \overline{(H - \overline{H})^2}$ and the mean square value δ^2 of the fractional fluctuation $(H - \overline{H})/\overline{H}$, i.e. $\delta^2 = \overline{\Delta H^2}/\overline{H}^2$.

28. Using the canonical distribution, find the following distributions (the various forms of the Maxwell distribution):

(1) the probability that the velocity of any particle of a system lies in the intervals $[v_x, v_x + dv_x]$, $[v_y, v_y + dv_y]$, $[v_z, v_z + dv_z]$;

(2) the probability that the speed of any particle lies in the interval $[v, v + dv]$;

(3) the probability that the kinetic energy of any particle lies in the interval $[\varepsilon, \varepsilon + d\varepsilon]$.

29. Using the results of Problem 28, find the following values:

(a) the mean of the nth power of the velocity $\overline{v^n}$ for $n > -2$;

(b) the mean speed $\overline{v}$ and the mean of the square of speed $\overline{v^2}$;

(c) the most probable speed of the particles v_0.

30. Find the average energy $\overline{\varepsilon}$ and the most probable kinetic energy ε_0 of a system of particles of Problem 28. Explain why they do not coincide.

31. The energy of a particle in a relativistic gas is linked with the momentum of the particle by the relationship $\varepsilon = c\,(m^2c^2 + p^2)^{1/2}$. Find the Maxwell distribution for this case.

32. How will the Maxwell distribution change if the system as a whole moves with a velocity **u**?

33. Find the probability that two particles will have the absolute value of the relative velocity $\mathbf{v}' = \mathbf{v}_1 - \mathbf{v}_2$ in the interval $[v', v' + dv']$. Also find $\overline{v'}$.

34. Find the total number of collisions of one molecule with the other ones in the system. Consider the molecules as being elastic spheres of radius R_0. Determine the mean free path λ.

35. Determine the ratio of the number of particles that have an energy greater than ε_1 to that with an energy less than ε_1 $(\varepsilon_1 = kT)$.

36. Determine the total scattering cross section as a function of temperature if the potential energy of interac-

tion is

$$U = \infty \qquad \text{for} \quad r \leqslant R_0,$$

$$= -\frac{\alpha}{r^n} \quad \text{for} \quad r > R_0 \quad (n > 2, \ \alpha > 0).$$

37. Determine the mean number of collisions experienced by a single molecule with other molecules in one second in the two-dimensional case (i.e. for a surface).

38. Each atom of a gas radiates monochromatic light of wavelength λ_0 and intensity J_0. Find the radiation intensity of the gas as a whole as a function of λ, the gas consisting of N atoms.

39. Assuming that the potential energy of an electron inside a metal is less than its energy outside the metal by $W = e\varphi$, determine the current density of thermionic emission. The concentration of electrons in the metal is n_0, and the electron mass is m.

40. Prove that for a perfect gas with the known law $\varepsilon = \varepsilon(\mathbf{p})$ the pressure is given by the formula

$$p = \frac{4\pi}{3} \frac{N}{V} \int_0^\infty \frac{\partial \varepsilon}{\partial |\mathbf{p}|} |\mathbf{p}|^3 f(\mathbf{p}) \, d|\mathbf{p}|$$

where $f(\mathbf{p})$ is the momentum distribution function [i.e. $f(\mathbf{p})$ is the probability that the particle has momentum $\mathbf{p}$].

41. Using the canonical distribution, find for a perfect gas contained in an external potential force field $u(x, y, z)$ the probability that the coordinates of any particle of the gas will lie in the intervals $[x, x+dx]$, $[y, y+dy]$, $[z, z+dz]$.

42. Find the centre of gravity of a column of perfect gas in a homogeneous gravitational field if the acceleration of gravity is g, the mass of one molecule is m, and the temperature is T.

43. A mixture of l perfect gases consisting of equal numbers of particles but with different masses of atoms $m_1, \ldots, m_k, \ldots, m_l$ is contained in a cylinder of height h and radius R and is placed in the earth's gravitational field. Determine the centre of gravity of the mixture.

44. Let the quantity $4\pi v^2 f(v^2)\, dv$ represent the probability that the velocity of a molecule lies in the interval

$[v, v + dv]$, with $f(v^2)$ being a differentiable function of unknown form. Find the Maxwell velocity distribution assuming that the probability distributions for the three Cartesian components of velocity are (1) independent; (2) identical.

45. Two vessels in which pressures and temperatures are maintained at p_1, T_1 and p_2, T_2, respectively, are connected by a short pipe with a cross section S. Determine the mass of a gas flowing from one vessel into the other if the mass of the gas molecules is m and if $p_1 = 2p_2$ and $T_1 = 2T_2$.

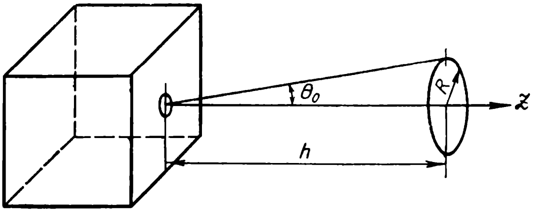

Fig. 46

46. A sphere of radius R moves with a velocity u in greatly rarefied gas (i.e. the mean free path of a particle in the gas is considerably greater than R). The temperature of the gas is T and its density is n_0. Assuming that the collisions of the particles of gas with the sphere are elastic, determine the force of resistance experienced by the sphere as it moves. Compare the result with Stokes' law for the force of resistance experienced by a similar sphere moving in viscous liquid.

47. A small round aperture with a cross section S has been made in a vessel containing perfect gas. Find the number of particles that fall on a round disc of radius R situated at a distance h from the aperture. The plane of the disc is parallel to the plane of the aperture (Fig. 46). The centres of S and the disc lie on a straight line perpendicular to the plane of the aperture. The molecules of gas are governed by the Maxwell velocity distribution.

48. A rarefied gas is contained in a vessel at a pressure p. Determine the velocity of outflow v of the gas into a vacuum

through a small aperture S_0 if the molecules of gas are governed by the Maxwell velocity distribution.

49. Determine the permittivity of a perfect gas consisting of N molecules with a dipole moment $\mathbf{p}_0$ each. The gas is located in an external homogeneous electric field $\mathbf{E}$.

50. Make the same calculations as in Problem 49 but take into account the polarizability α of the molecules, which does not depend on the magnitude of the external field.

51. Prove that a system of interacting electric charges (the problem is considered classical) cannot be in equilibrium in an external magnetic field.

52. A body with a potential φ_0 is placed into a plasma consisting of electrons $(-e)$ and ions $(+e)$. Determine the Debye screening distance assuming that the temperature of the electrons T_e differs from the temperature of the ions T_i and the plasma is quasi-neutral. The number of particles in unit volume is n_0.

53. A perfect gas is contained in a vessel that is closed by a movable piston loaded with mass M. Find the equation of state of the gas.

54. Derive the Dalton law for a mixture of n perfect gases:

$$p=\sum_{i=1}^{n} p_i$$

where p_i is the partial pressure of the ith component.

55. Prove that for any system with a Hamiltonian H,

$$C_V=\frac{1}{kT^2}\overline{(H-\overline{H})^2}.$$

56. Prove that for any physical quantity $f(q_1, \ldots, q_N, p_1, \ldots, p_N)$,

$$\overline{f\frac{\partial H}{\partial q_i}}=kT\left(\overline{\frac{\partial f}{\partial q_i}}\right) \quad \text{and} \quad \overline{f\frac{\partial H}{\partial p_i}}=kT\left(\overline{\frac{\partial f}{\partial p_i}}\right).$$

57. Particles of a rarefied gas interact according to the law (Fig. 47)

$$\begin{aligned} U &= \infty && \text{for} \quad r\leqslant r_0 \\ &= -U_0\left(\frac{r_0}{r}\right)^6 && \text{for} \quad r>r_0 \end{aligned}$$

Find the heat capacity for the gas.

58. Show that the van der Waals equation can be written as

$$\left(\pi + \frac{3}{\omega^2}\right)(3\omega - 1) = 8\tau$$

(the reduced van der Waals equation), where

$$\pi = \frac{p}{p_{cr}}, \quad \tau = \frac{T}{T_{cr}}, \quad \omega = \frac{V}{V_{cr}}$$

(p_{cr}, T_{cr}, V_{cr} are the critical pressure, temperature and volume, respectively).

59. Determine the average energy and the heat capacity C_V of a perfect gas consisting of N diatomic molecules, taking account of the anharmonicity of atomic vibrations in the molecules. Examine the case of low temperatures.

60. Atoms in a diatomic molecule interact according to the law

$$U(r) = \frac{A}{r^{12}} - \frac{B}{r^6} \quad (A, B > 0)$$

(the so-called Lennard-Jones "12-6" potential). Find the coefficient of linear expansion for such a molecule.

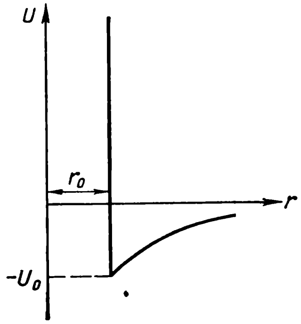

Fig. 47

61. For a system consisting of a large number of particles the heat capacity is

$$C_V = \alpha T^n \quad (\alpha > 0, \ n > 1)$$

Find $\Omega(E)$ for such a system.

62. Entropy is sometimes defined as $S = k \ln \Gamma(E)$ or as $S = k \ln \Omega(E)$. Prove the equivalence of these definitions for systems with a large number of particles.

63. Using the general properties of entropy and probability and assuming that there is a functional dependence between entropy and probability, prove the Boltzmann relationship

$$S = k \ln w.$$

64. Find the work that has to be spent to polarize a unit volume of an isotropic dielectric.

65. If each of the three variables A, B, C is a differentiable function of the other two, regarded as independent, prove that

(a) $\left(\frac{\partial A}{\partial B}\right)_C \left(\frac{\partial B}{\partial C}\right)_A \left(\frac{\partial C}{\partial A}\right)_B = -1;$

(b) $\left(\frac{\partial A}{\partial C}\right)_B = 1 \Big/ \left(\frac{\partial C}{\partial A}\right)_B.$

66. Establish the connection between the thermal coefficients

$$\alpha = \frac{1}{V_0}\left(\frac{\partial V}{\partial T}\right)_p, \quad \beta = -\frac{1}{V_0}\left(\frac{\partial V}{\partial p}\right)_T, \quad \gamma = \frac{1}{p_0}\left(\frac{\partial p}{\partial T}\right)_V$$

where V_0 and p_0 are, respectively, the mean volume and mean pressure in an arbitrary thermodynamic system.

67. Write the Dieterici equation of state for a real gas in reduced variables (see Problem 58):

$$p = \frac{RT}{V-b} \exp\left(-\frac{a}{RTV}\right)$$

Do the same for the equation of state

$$\left(p + \frac{a}{V^{5/3}}\right)(V-b) = RT.$$

68. Determine the Boyle temperature for real gases (i.e. the temperature at which the second virial coefficient is zero) if a and b are given on the basis of the van der Waals and Dieterici equations of state.

69. Determine the velocity of a sound wave propagating in a real gas that satisfies the van der Waals equation of state.

70. Find the adiabatic equation for an ideal paramagnetic substance (see Problem 49).

71. Find the adiabatic equation for a real gas.

72. Calculate the difference between the heat capacities of a dielectric with a constant E (the electric field intensity) and with a constant D (the electric displacement), i.e. $C_E - C_D$.

73. Show that $\left(\frac{\partial C_V}{\partial V}\right)_T = T\left(\frac{\partial^2 p}{\partial T^2}\right)_V.$

74. Determine the efficiency of two thermodynamic engines working in cyclic processes depicted in Figs. 48 and 49 if the parameters of the cycle $\varepsilon = \frac{V_1}{V_2}$ and $\rho = \frac{V_3}{V_2}$ are given and the working fluid is a perfect gas.

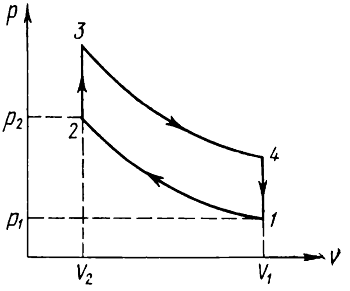

Fig. 48

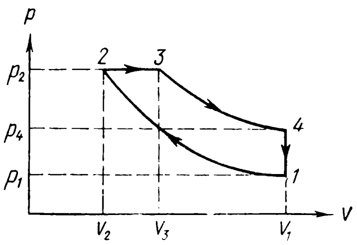

Fig. 49

75. Using the first and second laws of thermodynamics, prove the following relationships

$$\frac{C_p}{C_V} = \frac{\left(\frac{\partial V}{\partial p}\right)_T}{\left(\frac{\partial V}{\partial p}\right)_S} = \frac{\beta}{\delta}, \quad C_V = \frac{TV_0\alpha^2\delta}{(\beta-\delta)\,\beta}, \quad C_p = \frac{TV_0\alpha^2}{\beta-\delta}$$

where $\alpha = \frac{1}{V_0}\left(\frac{\partial V}{\partial T}\right)_p$ is the coefficient of volume expansion, $\beta = -\frac{1}{V_0}\left(\frac{\partial V}{\partial p}\right)_T$ is the isothermal compressibility, and $\delta = -\frac{1}{V_0}\left(\frac{\partial V}{\partial p}\right)_S$ is the adiabatic compressibility.

76. Determine the temperature dependence of the e.m.f. of a galvanic cell in the differential form.

77. Prove the relationship [see formula (25) of Appendix 4]

$$\left(\frac{\partial T}{\partial V}\right)_E = \frac{p - T\left(\frac{\partial p}{\partial T}\right)_V}{C_V}.$$

78. Find the average energy of a rarefied plasma that occupies a volume V. The plasma is a system of two kinds of

oppositely charged particles (N particles of each kind with charges $+e$ and $-e$).

79. Use the result of Problem 78 to determine p, S, and C_V for the plasma.

80. Find $C_p - C_V$ in (a) V, T-and (b) p, T-variables.

81. Show that

$$C_p - C_V = \left[V - \left(\frac{\partial H}{\partial p}\right)_T\right]\left(\frac{\partial p}{\partial T}\right)_V$$

where $H = E + pV$ is the enthalpy of the system.

82. Prove that for an isotropic dielectric in an external electric field $\mathbf{E}$,

$$dF = -S\,dT - p\,dV - (\mathbf{P}\cdot d\mathbf{E})$$

where $\mathbf{P}$ is the polarization vector for the dielectric.

83. Find $C_p - C_V$ for a van der Waals gas.

84. What amount of heat must be transferred to one mole of a real gas for it to expand from volume V_1 to volume V_2 if pressure p is constant. The equation of state is

$$\left(p + \frac{a}{V^2}\right)(V - b) = RT.$$

85. A process in a system is called polytropic if it does not change the system's heat capacity $C = dQ/dT$. Using this definition, find the equation of such a process for a monatomic perfect gas.

86. Determine the entropy of a gas whose equations of state are

$$V = V_0[1 + \alpha(T - T_0)], \quad \left(\frac{\partial V}{\partial p}\right)_T = 0; \quad C_p = \text{constant}.$$

87. Find the adiabatic equation of a gas whose equation of state is

$$p = p_0(1 + \alpha T - \beta V); \quad C_V = \text{constant}.$$

88. Determine the difference between the energy release of the reaction of formation of one gram-mole of water vapour at constant pressure and the energy release of the same reaction if it takes place without the performance of external work.

89. Determine the efficiency of a thermodynamic engine working along the Carnot cycle if the state of the working

fluid is given by the equations

$$V = V_0 [1 + \alpha (T - T_0)], \quad \left(\frac{\partial V}{\partial p} \right)_T = 0.$$

90. Show that when heat is transferred from a heated body to a less heated one, the entropy increases. Assume that the temperatures of the bodies equalize and the heat capacities do not depend on temperature.

91. Find the change in the entropy of a body when it expands isobarically.

92. Prove that an isotherm cannot intersect an adiabat twice.

93. In the temperature interval from 0 to 4 °C the coefficient of volume expansion of water is negative. Show that in this interval water cools under adiabatic compression.

94. Can the relationship $C_p = C_V$ hold for water?

95. The main reason for the fall in temperature in the atmosphere with growing altitude is the adiabatic expansion of the upcurrent. Using the adiabatic equation of a perfect gas, find the change in temperature with altitude.

96. Use the first law of thermodynamics to show that atmospheric air with a temperature gradient less or greater than the gradient found in Problem 95 will be, respectively, stable or unstable in relation to convection.

97. Determine the energy release in the polarization of a unit volume of a dielectric, neglecting the change in its specific volume and assuming that

$$\mathbf{p} = \frac{\varepsilon (T) - 1}{4\pi} \mathbf{E}.$$

98. Show that for a magnetic sample placed in an external magnetic field **H**, given the condition that E_0 is independent of **H** (E_0 is the internal energy of the sample in a vacuum neglecting the energy of the field there), the following relation holds for magnetization:

$$M = f\left(\frac{H}{T} \right).$$

99. Find the entropy and internal energy of a substance in an electric field using the following expression for the

Helmholtz free energy

$$F = F_{E=0} - \frac{\varepsilon - 1}{8\pi} V\mathbf{E}^2.$$

100. Determine the expressions for C_V, F, S, H, Φ, C_p for equilibrium radiation.

101. Using the grand canonical distribution, find the expressions for μ, p, and S for a monatomic perfect gas.

102. Using the grand canonical distribution, show that the Poisson distribution describes a system of N noninteracting particles.

103. Find the chemical potential of a perfect gas in an external potential field $U = U(x, y, z)$.

104. Particles with spin are in an external homogeneous magnetic field **B**. What is the ratio of the number of particles with spins along the field to the number of particles with spins in the opposite direction?

105. Using the properties of the grand canonical distribution, show that

$$pV = kT \ln Z$$

where Z is the grand partition function.

106. Prove that $\overline{N} = V\left(\frac{\partial p}{\partial \mu}\right)_{T,\,V}$ and $\mu = \left(\frac{\partial E}{\partial N}\right)_{S,\,V} = \left(\frac{\partial H}{\partial N}\right)_{S,\,p}$.

107. Find C_V expressed in terms of T, μ, and V.

108. Assume that the latent heat λ is constant and determine the temperature dependence of the saturated vapour pressure if the vapour is in equilibrium with the solid phase.

109. A solution contains N molecules of solvent and n molecules of the solute ($n \ll N$). Determine the chemical potentials of the solution μ and the solute μ_1 if the chemical potential of the solvent is μ_0.

110. A solution with a concentration $c \ll 1$ of the solute is in a homogeneous gravitational field. Determine the change in the concentration c with altitude.

111. Show that the chemical potential of black-body radiation is zero.

112. Using the conditions of stability of equilibrium, prove that when a constraint is applied to a dynamic sys-

tem in equilibrium, a change takes place within the system, opposing the constraint and tending to restore equilibrium (the Le Chatelier-Braun principle).

113. Determine the condition for mechanical equilibrium of an isolated system consisting of a spherical liquid drop of radius R and its surrounding vapour.

114. In the following reaction of gases,

$$H_2O + CO \rightleftarrows CO_2 + H_2$$

the equilibrium set in when the temperature was T_0 and the proportions were: for CO_2, m_1; for CO, m_2; for H_2O, m_3; and for H_2, m_4 (the proportions are given in moles). Determine the affinity constant $K(p, T)$.

115. Calculate the critical value of the radius of a drop of liquid when steam condenses.

116. Prove that a charged drop of liquid will grow even in an unsaturated vapour.

117. An equilibrium reaction $A \rightleftarrows I^+ + e^-$ (thermal ionization) takes place in a system. Assuming the temperatures of the ions and electrons to be the same, determine the degree of single ionization as a function of T and p. The gases are considered perfect and the energy of ionization is ε_0.

118. Calculate the amount of heat given off during a chemical reaction in which p and T are constant. The affinity constant K is known.

119. Determine K and ΔQ per particle (see Problem 118) for the dissociation of diatomic hydrogen

$$H_2 \rightleftarrows H + H$$

The dissociation energy for a hydrogen molecule is $\Delta\varepsilon = 2\varepsilon_0^{H} - \varepsilon_0^{H_2}$.

120. Find the osmotic pressure between solutions of different concentrations that are separated by a semipermeable membrane.

121. Prove that the relationship $\Xi = -\frac{2}{3} E$ holds for free particles with energy $\varepsilon = p^2/2m$ that obey the Fermi-Dirac or Bose-Einstein statistics.

122. A system can be in two quantum states with energies 0 and ε. The degeneracy multiplicities of the states are g_1

and g_2. Find the dependence of S on E and analyze this dependence.

123. Determine the heat capacity of a system of N independent, two-dimensional harmonic oscillators each of which has $(n+1)$-fold degenerate energy levels

$$\varepsilon_n = (n+1)\,\hbar\omega \quad (n = 0, 1, \ldots).$$

124. A system has a nondegenerate energy spectrum $\varepsilon_l = l\varepsilon$ $(l = 0, 1, 2, \ldots, n-1)$. Determine the average energy of such a system.

125. Represent the elastic vibrations of a solid in the Debye model as a phonon gas that obeys the Bose-Einstein statistics and find its energy. The volume of the body is V, and the velocity of propagation of transverse and longitudinal waves is c_t and c_l, respectively. Examine the case of low temperatures.

126. Determine the heat capacity of an ultrarelativistic degenerate electron gas $(\varepsilon = cp)$.

127. Find the thermionic emission current provided the electrons obey the Fermi-Dirac statistics and the work function for an electron escaping from the metal is W. Assume that $W - \mu \gg kT$, where μ is the chemical potential level.

128. Find the pressure in an electron gas at $T = 0$ using the Fermi distribution.

129. Determine the pressure in a degenerate electron gas for $\frac{kT}{\mu} \ll 1$.

130. Using the grand canonical distribution, find the dependence of entropy S on $\bar{n}_i$ for an ideal gas that obeys (1) the Fermi-Dirac statistics; (2) the Bose-Einstein statistics.

131. Let $g(\varepsilon)$ be the single-particle density of states. Show that the heat capacity of a gas that obeys the Fermi-Dirac statistics for $kT \ll \mu_0$ is given as

$$C_V = \frac{\pi^2}{3} k^2 T g(\mu_0).$$

132. Denoting the energy of a system as $E = \sum_i \bar{n}_i \varepsilon_i$, where ε_i is the energy level of the ith state of the system and $\bar{n}_i$ is the mean population of the ith state, explain the meaning of $\overline{dA}$ and dQ in reversible processes.

133. In the simplest case the spin waves in ferromagnetic substances satisfy a dispersion equation $\omega = Ak^2$, where $\mathbf{k}$ is the wave vector in the spin wave, and A is a constant. Determine what these excitations contribute to the heat capacity of crystals at low temperatures.

134. Prove that for crystals the Mie-Grüneisen relation holds true, i.e.

$$\alpha_l = \frac{\beta\gamma}{3V} C_V$$

where $\alpha_l = \frac{1}{3V}\left(\frac{\partial V}{\partial T}\right)_p$ is the coefficient of linear expansion, $\beta = -\frac{1}{V}\left(\frac{\partial V}{\partial p}\right)_T$ the isothermal compressibility, and $\gamma = -\frac{\partial \ln \nu_j}{\partial \ln V}$ is assumed to have the same value for all normal modes ν_j.

135. Determine $\Omega(E)$ for a solid at $T \ll \frac{\hbar\omega_{\max}}{k}$.

136. Find the Fermi level in an intrinsic nondegenerate semiconductor if the width of the forbidden band has a temperature dependence of the form

$$E_g = E_g^0 - \zeta T \quad (\zeta > 0).$$

137. Determine the concentration of electrons in germanium if the dependence of energy on the wave vector (the dispersion equation) has the form

$$E_\alpha(\mathbf{k}) = E_c + \frac{\hbar^2 (k_x - k_x^\alpha)^2}{2m_t} + \frac{\hbar^2 (k_y - k_y^\alpha)^2}{2m_t} + \frac{\hbar^2 (k_z - k_z^\alpha)^2}{2m_l}$$

where $\alpha = 1, 2, 3, 4$ is the number of equivalent minimums in the conduction band, and m_t and m_l are, respectively, the transverse and longitudinal masses of the electron.

138. In the conduction band of gallium arsenide the dispersion equation is of the form shown in Fig. 50. Find the Fermi level for two extreme cases:

(a) a nondegenerate electron gas;

(b) a highly degenerate electron gas.

139. Find the concentration of electrons in a semiconductor if it is known that, when the magnitude of the wave

vector **k** is small, the dispersion equation has the form

$$E(\mathbf{k}) = E_0 + \frac{\hbar^2 k^2}{2m_n}(1 - \gamma k^2)$$

where γ is a constant.

140. Determine the concentration of electrons in a semiconductor of the n-type with a narrow forbidden band (in-

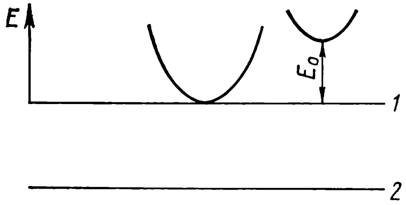

Fig. 50

dium antimonide). The dispersion equation for this case is

$$E(\mathbf{k}) = E_0 + \frac{1}{2}\left(\pm\sqrt{E_g^2 + \frac{2\hbar k^2 E_g}{m(0)}} - E_g\right)$$

where E_g is the width of the forbidden band, and $m(0)$ is the effective mass of the electron near the edge of the band $[m(0) \ll m_0]$.

141. Consider a semiconductor with a width of the forbidden band E_g and the donor and acceptor levels lying at a distance of E_1 and E_2 from the lower edge of the conduction band. Assuming that the electrons in the conduction band and the holes in the valence band obey classical statistics, find the chemical potential for this semiconductor. The acceptor concentration is n_2, and the donor concentration is n_1. The effective mass of the electron is m_n, and that of the hole is m_p. Consider the case of a donor semiconductor ($n_2 = 0$, $E_g \gg E_1$).

142. Prove that for an extrinsic semiconductor the product of the hole concentration to the electron concentration is proportional to the square of the electron (or hole) concentration in an intrinsic semiconductor.

143. Determine the heat capacities of electrons and holes for an intrinsic nondegenerate semiconductor with the width of the forbidden band being E_0.

144. Find the equation of state for a completely degenerate relativistic electron gas with an energy $\varepsilon = c\,(m^2c^2 + p^2)^{1/2}$.

145. What is the equilibrium ratio of the concentrations of ortho- and parahydrogen at a temperature of $T \ll T_c = \frac{\hbar^2}{8\pi^2 Ik}$, where I is the moment of inertia of a hydrogen molecule?

146. Derive the Fermi-Dirac distribution from an examination of particle collision, allowing for the Pauli exclusion principle.

147. For a crystal in the Debye model determine the mean square displacement of the atoms of the lattice. A unit cell of the crystal contains one atom.

148. How do the thermodynamic variables, density and temperature, for instance, transform in passing from one inertial frame of reference, K, to another frame, K'?

149. Derive the Planck formula for thermal radiation in a medium with dispersion $n = n(\nu)$ (n is the index of refraction).

150. Find the spin paramagnetic susceptibility of a system of free electrons if the single-electron density of states per unit volume is $g(E)$. Consider the case of a weak field.

151. Determine the magnetic susceptibility of a nondegenerate electron gas. The energy levels of an electron moving in a magnetic field $B = B_z$ are given by the formula

$$\varepsilon = \frac{1}{2m_n} p^2 + 2\mu^* B\,(l + 1/2) \pm \mu_B B \quad (l = 0, 1, \ldots)$$

where m_n is the effective mass of the electron, and $\mu_B = \frac{e\hbar}{2mc}$ is the Bohr magneton.

152. Determine the orbital magnetic susceptibility of a degenerate electron gas in a weak magnetic field ($\mu_B B \ll kT$).

153. Determine the sensitivity limit of a mirror-type galvanometer. The torsion constant of the suspension is α.

154. Find the fluctuations of pressure in a homogeneous system that is placed in a heat bath and show that for the system to be in stable equilibrium the following condition

must hold:

$$\left(\frac{\partial p}{\partial V}\right)_S < 0.$$

155. Determine the decrease in entropy that develops during free oscillations of a simple pendulum.

156. Find the correlations $\overline{\Delta T \Delta p}$ and $\overline{\Delta S \Delta V}$.

157. Show that $\overline{\Delta S^2} = kC_p$ and $\overline{\Delta S \Delta p} = 0$.

158. Find the energy fluctuations for a system with two energy levels.

159. Determine the fluctuations of the number of particles in perfect gases that obey (a) the Boltzmann distribution; (b) the Fermi-Dirac distribution; and (c) the Bose-Einstein distribution. Analyze the results obtained.

160. Show that the following relationship holds for the grand canonical ensemble:

$$\overline{\Delta H^2} = k\left(\frac{\partial E}{\partial T}\right)_{\overline{N}} + \overline{\Delta N^2}\left(\frac{\partial E}{\partial \overline{N}}\right)_T^2.$$

161. Calculate the fluctuations of the position of the centre of mass for a homogeneous perfect gas contained in a spherical vessel of radius R_0.

162. Let q_i and q_k be the generalized coordinates of particles and α_i and α_k the additional generalized forces switched on at the initial time and acting along the directions q_i and q_k. Prove that

$$\overline{(q^t - q^0)^2} = 2kT\left[\frac{\partial q^t}{\partial \alpha}\right]_{\alpha=0}.$$

163. Calculate the mean square displacement of a Brownian particle of mass m and radius a, moving in a viscous medium with a coefficient of viscosity η.

164. Find the value of the Avogadro number N_A if the mean square displacement for a Brownian particle of mass m and radius a in time t is $\overline{\Delta x^2}$.

165. Find $\overline{(z - z_0)^2}$ for a Brownian particle in the gravitational field of the earth.

166. Particles diffuse through a one-dimensional potential barrier $U(x)$ in a stationary flow. Find the flux density if the densities of the number of particles in cross sections x_1 and x_2 are known (the Kramers formula).

167. Assuming that the Boltzmann equation for a highly degenerate gas is

$$\frac{\partial f}{\partial t}+\mathbf{v}\frac{\partial f}{\partial \mathbf{r}}-e\mathbf{E}\frac{\partial f}{\partial \mathbf{p}}=-\frac{f-f_0}{\tau}$$

where τ is the relaxation time, find the electric conductivity of metals at low temperatures.

168. For a nondegenerate electron gas find the electrical and thermal conductivities assuming that along the x-axis there is a constant temperature gradient, $k\frac{\partial T}{\partial x}=$ constant, an electric field $\mathbf{E}$ is applied to the system, and that $\tau = Av^l$ $(A > 0,\ l > -7)$.

169. Determine the tensor of electrical conductivity for the electrons in a metal that is placed in homogeneous electric and magnetic fields. Consider the electron gas to be degenerate.

170. Find the coefficient of viscosity for a gas flow with a velocity gradient β along the x-axis. The gas obeys classical statistics.

171. Particles of mass m are distributed with a constant density inside a sphere of radius R. The temperature inside the sphere is T. At time $t = 0$ the sphere disappears and the particles fly in all directions. Neglecting the collisions between the particles, find the density of the cloud of particles as a function of time and position.

Answers

SECTION I

1. A parabola with the parameter $\frac{mv_0^2}{eE}$.

2. $T = \frac{2mv_0}{eE}$.

3. A circle of radius v_0/ω_H, where $\omega_H = \frac{e\mu_0 H}{m}$ is the cyclotron frequency.

4. $z = \frac{eE_0}{m\omega^2}\left[\left(1-\cos\frac{\omega x}{v_0}\right)\cos\omega t_0 - \left(\frac{\omega x}{v_0} - \sin\frac{\omega x}{v_0}\right)\times\right.$ $\left.\times \sin\omega t_0\right]$; the z-axis is directed along the field, and t_0 is the instant of time when the particle enters the field.

5. In a system of coordinates with the y-axis parallel to **H** and the x-axis perpendicular to the plane containing vectors **E** and **H**, the path is

$$x = -\frac{v_{0z}}{\omega_H}(1-\cos\omega_H t) + \left(\frac{v_{0x}}{\omega_H} + \frac{eE}{m\omega_H^2}\sin\alpha\right)\sin\omega_H t - \frac{eEt}{m\omega_H}\sin\alpha$$

$$y = \frac{eEt^2}{2m}\cos\alpha + v_{0y}t$$

$$z = \frac{v_{0z}}{\omega_H}\sin\omega_H t + \left(\frac{v_{0x}}{\omega_H} + \frac{eE}{m\omega_H^2}\sin\alpha\right)\cdot(1-\cos\omega_H t)$$

(at $t=0$ the particle was in the origin of coordinates).

6. Put $\alpha=0$, $v_{0y}=v_{0z}=0$, $v_{0x}=v_0$ in the solution of Problem 5, and find $x=\frac{v_0}{\omega_H}\sin\omega_H t$. By setting $x=l$, we can determine T from the relationship $\frac{\omega_H l}{v_0} = \sin\omega_H T \approx$ $\approx \omega_H T$ $\left(\text{since } \frac{\omega_H l}{v_0} \ll 1\right)$. Substituting $T = l/v_0$ for t in the

expressions for y and z and eliminating v_0, we find that $y = 2pz^2$, where $p = \frac{eE}{m\omega_H^2 l^2}$.

7. (1) $l \geqslant \frac{v_0}{|\omega_H|}\sqrt{\cos^2\alpha\sin^2\beta + \sin^2\alpha} + \frac{v_0}{\omega_H}\cos\alpha\sin\beta$.

Hints to questions (2) and (3): the equation of the projection of the path on the xy-plane has the form

$$\left(y + \frac{v_0}{\omega_H}\sin\alpha\right)^2 + \left(x - \frac{v_0}{\omega_H}\cos\alpha\sin\beta\right)^2 = \frac{v_0^2}{\omega_H^2}(\sin^2\alpha + \cos^2\alpha\sin^2\beta)$$

Using this equation, we can find the points where the particle comes out of the magnetic wall and construct the tangents to the path at these points.

8. *Hint*. First find the coordinates of a specific electron as a function of time t and of the instant t_0 when the electron enters the space between the plates. Next find the instant when the electron hits screen S and the electron's coordinates at that instant as functions of t_0. This gives the equation of the path of the beam of electrons on the screen in the following form:

$$x = -\frac{la_1}{\omega v_0}\left[\cos\omega\left(t_0 + \frac{l}{v_0}\right) - 2\cos\omega t_0\right] - \frac{a_1}{\omega^2}\left[\sin\omega\left(t_0 + \frac{l}{v_0}\right) - \sin\omega t_0\right] = A\cos\omega t_0 + B\sin\omega t_0.$$

$$y = \frac{la_2}{\omega v_0}\left[\sin\omega\left(t_0 + \frac{l}{v_0}\right) - 2\sin\omega t_0\right] - \frac{a_2}{\omega^2}\left[\cos\omega\left(t_0 + \frac{l}{v_0}\right) - \cos\omega t_0\right] = C\sin\omega t_0 + D\cos\omega t_0$$

where $a_1 = \frac{eV_1}{md}$, $a_2 = \frac{eV_2}{md}$; d is the distance between the plates.

The path on the screen is the ellipse

$$(Ay - Dx)^2 + (By - Cx)^2 = (AC - BD)^2.$$

9. $x = Ae^{i\omega_1 t} + Be^{i\omega_2 t} + Ce^{-i\omega_1 t} + De^{-i\omega_2 t}$

$$y = Ae^{i\omega_1\left(t-\frac{\pi}{2}\right)} - Be^{i\omega_2\left(t+\frac{\pi}{2}\right)} + Ce^{-i\omega_1\left(t-\frac{\pi}{2}\right)} - De^{-i\omega_2\left(t-\frac{\pi}{2}\right)}$$

$$z = A\cos(\omega_0 t + \varphi)$$

where ω_0 is the natural frequency of the oscillator $\omega_{1,2} = \frac{\omega_H}{2} \pm \sqrt{\omega_0^2 + \frac{\omega_H^2}{4}}$ (the z-axis is directed along the magnetic field), and $\omega_H = \frac{e\mu_0 H}{m} = \frac{eB}{m}$ (μ_0 is the permeability of empty space).

10. $x = C_1 + C_2 e^{-\frac{\gamma}{m}t} + \frac{eE}{\gamma}t$

$$y = C_3 + C_4 e^{-\frac{\gamma}{m}t}$$

$$z = C_5 + C_6 e^{-\frac{\gamma}{m}t}$$

where the constants of integration $C_1, \ldots, C_6$ are determined from the initial conditions. The x-axis is directed along the electric field.

11. $x = C_1 + Ae^{-\frac{\gamma}{m}t}\cos(\omega_H t + \alpha)$

$$y = C_2 - Ae^{-\frac{\gamma}{m}t}\sin(\omega_H t + \alpha)$$

$$z = C_3 + C_4 e^{-\frac{\gamma}{m}t}$$

The z-axis is directed along the magnetic field.

12. $T = \frac{h}{\sqrt{2g(H+h)}}\left[\frac{\pi}{2} + \arctan\frac{h}{2\sqrt{H(H+h)}}\right].$

13. $v = \sqrt{\frac{mg}{\gamma}}\tanh\sqrt{\frac{\gamma g}{m}}\,t,\quad x = \frac{m}{\gamma}\ln\cosh\sqrt{\frac{\gamma g}{m}}\,t,$

$$v \to \sqrt{\frac{mg}{\gamma}} \quad (t \to \infty).$$

14. $x = \frac{h}{6}\cos\omega t$, where $\omega = \sqrt{\frac{c + \pi\rho r^2 g}{M}}$.

15. $v=\sqrt{\frac{2gRh}{R+h}}, \quad T=\frac{1}{R}\sqrt{\frac{R+h}{2g}}\left(\sqrt{Rh}\right.$
$$\left.+\frac{R+h}{2}\arccos\frac{R-h}{R+h}\right).$$

16. $h=\frac{mv_0\sin\alpha}{\gamma}-\frac{m^2g}{\gamma^2}\ln\left(1+\frac{\gamma v_0}{mg}\sin\alpha\right)$

$$x=\frac{mv\cos\alpha}{\gamma}(1-e^{-\gamma t/m})$$

$$y=\frac{m}{\gamma}\left(v_0\sin\alpha+\frac{mg}{\gamma}\right)(1-e^{-\gamma t/m})-\frac{mgt}{\gamma}$$

$$s=\frac{v_0^2\sin 2\alpha}{2g\left(1+\frac{mv_0}{g\gamma}\sin\alpha\right)}.$$

17. $\mathbf{F}=-m\omega^2\mathbf{r}.$

18. $F_x=0, \quad F_y=-\frac{mv_0^2b^4}{a^2y^3}.$

19. $x=be^{-\alpha t}\cos(\omega t+\varphi)$

where $b=\frac{f_0/m}{\sqrt{(\omega_0^2+\alpha^2-\omega^2-2\alpha\omega)^2+4\omega^2(\omega-\alpha)^2}}$,

$$\tan\varphi=-\frac{2\omega(\omega-\alpha)}{\omega_0^2-\omega^2+\alpha^2-2\alpha\omega}.$$

20. The product $x_0\times\dot{x}_0$ must be less than zero, i.e. the initial velocity $\dot{x}_0$ must be directed opposite to the initial displacement x_0. In addition, the following conditions must hold:

$$\left|\frac{\dot{x}_0}{x_0}\right|>\gamma-\sqrt{\gamma^2-\omega_0^2}, \quad \gamma\geqslant\omega_0.$$

21. $\varphi=\frac{RI_0t^2}{4I}\sqrt{\frac{mV_0}{2e}}$, where $I=2\rho dlR^3$ is the moment of inertia of the vanes (for $d\ll l$).

22. The space charge can be taken into account by using the Langmuir-Boguslavski formula (for our case the result was obtained by C. D. Child in 1911). This formula is also known as the "three halves power law". (1) The reaction force is $F=\frac{dM}{dt}v$, where $M=\frac{m}{e}Q$ is the total

mass of the particles, Q the total charge corresponding to that mass, and $v=\left(\frac{2eV}{m}\right)^{1/2}$. Then $F=\frac{m}{e}\left(\frac{2eV}{m}\right)^{1/2}\times$ $\times\frac{dQ}{dt}=\frac{m}{e}\left(\frac{2eV}{m}\right)^{1/2}jS$, where S is the area of plane K. Substituting the Langmuir-Boguslavski formula $j=$ $=\frac{8}{9}\varepsilon_0\left(\frac{e}{2m}\right)^{1/2}\frac{V^{3/2}}{d^2}$, we obtain the final result: $F=$ $=\frac{8}{9}\varepsilon_0 V^2 S/d^2$. (2) The power requirement is $P=IV=$ $=jSV=\frac{8}{9}\varepsilon_0\left(\frac{e}{2m}\right)^{1/2}\times V^{5/2}S/d^2$.

23. On the midline of the capacitor the following condition must hold:

$$\frac{4T}{r_1+r_2}=\frac{2eV}{(r_1+r_2)\ln\frac{r_2}{r_1}}+e\mu_0 H\sqrt{\frac{2T}{m}}.$$

26. $T=2\pi a^{3/2}\sqrt{\frac{1}{GM_{\mathrm{E}}}}=\pi GmM_{\mathrm{E}}\sqrt{\frac{m}{2|E|^3}}=2\pi\frac{a^{3/2}}{R_{\mathrm{E}}g^{1/2}}$, where M_{E} is the mass of the earth, R_{E} is its radius, and m is the mass of the satellite.

27. The equation of the path in polar coordinates is

$$\rho=\frac{p}{1+e\cos\sigma\theta}$$

where $p=\frac{l^2-2\beta}{\alpha}$, $e=\sqrt{1+\frac{2E}{\alpha^2}(l^2-2\beta)}$, and $\sigma=$ $=\sqrt{1-2\beta/l^2}$ (l is the ratio of the angular momentum to the mass).

(1) The particle will "fall" on the centre if $2\beta>l^2$, i.e. if σ is imaginary. The equation of the path is

$$\rho=\frac{|p|}{1+e\cosh|\sigma|\theta}$$

(2) The particle will scatter if $e\geqslant 1$.

(3) The particle will be in alternating motion if $\sigma=m/n$, where m and n are integers.

28. $\Phi=\pi-\frac{2}{\sigma}\arccos\left(-\frac{1}{e}\right)$, $l=sv_\infty$. These formulas together with the definition of e, $e=\sqrt{1+\frac{2E}{\alpha^2}(l^2-2\beta)}$,

give the relation between the impact parameter s and the scattering angle Φ.

29. $T=\sqrt{2m}\int\limits_{x_1}^{x_2}\frac{dx}{\sqrt{E-U(x)}}$; the motion takes place between the points x_1 and x_2 ("points of retrogression") which are the solution of the equation $U(x)=E$.

30. $T=\frac{\pi}{\alpha}\sqrt{\frac{2m}{|E|}}$.

31. $T=\frac{\pi\sqrt{2m}}{\alpha\sqrt{E+U_0}}$.

32. $\frac{1}{r}=\sqrt{\frac{2mE}{L^2+2m\alpha}}\cos\sigma\varphi$, where $\sigma=\sqrt{1+\frac{\alpha}{s^2E}}$ (L is the angular momentum and $s=L/\sqrt{2mE}$ the impact parameter).

33. $\theta=\pi\left[1-\frac{1}{\sqrt{1+\frac{\alpha}{s^2E}}}\right]$, $\sigma(\theta)=\frac{\alpha\pi^2}{E}$

$$\times\frac{\pi-\theta}{\theta^2(2\pi-\theta)\sin\theta}.$$

35. $v=v_0+u_1\ln\frac{m_0}{m(t)}$, where m_0 and v_0 are respectively the take-off mass and velocity.

36. For the earth $v_1=\sqrt{gR_E}\approx 8$ km/s, $v_2=v_1\sqrt{2}=$ $=\sqrt{2gR_E}\approx 11$ km/s;

for the moon $v_1=\sqrt{Gm_M/R_M}\approx 1.7$ km/s, $v_2=$ $=v_1\sqrt{2}\approx 2.4$ km/s, where $m_M=7.33\times 10^{25}$ g is the moon's mass, $R_M=1738$ km its radius, and $G=6.67\times 10^{-8}$ cm^{3}g^{-1}s^{-2} is the gravitational constant.

37. (1) $A=\frac{2F_0}{mT\omega^3}\sin\frac{\omega T}{2}$;

(2) $A=\frac{2F_0}{m\omega^2}\sin\frac{\omega T}{2}$;

(3) $A=\frac{F_0}{mT\omega^3}\sqrt{\omega^2T^2-2\omega T\sin\omega T+2(1-\cos\omega T)}$;

(4) $A=\frac{\pi F_0}{m\omega^2}$.

38. $\tan\theta = \frac{\sin\Phi}{m_1/m_2+\cos\Phi}$, $\qquad \Delta E = \frac{4m_1m_2}{(m_1+m_2)^2} E_1 \sin^2\frac{\Phi}{2}$, $m_1/m_2 = 1$.

39. $(x/r_0)^2 - k(y/v_0) = 1$: a hyperbola if the external force is repulsive $(k > 0)$; an ellipse if the force is attractive $(k < 0)$.

40. $x = \frac{h}{6}\sqrt{\frac{k^2}{k^2-\sigma^2}}\, e^{-\sigma t}\sin(\sqrt{k^2-\sigma^2}t+\varphi)$ if $k > \sigma$;

$$k^2 = \frac{c}{M} + \frac{\pi r^2 \rho g}{M}, \quad \sigma = \frac{\alpha}{2M}, \quad \tan\varphi = \sqrt{\frac{k^2-\sigma^2}{\sigma^2}}.$$

43. $\mathscr{L} = \frac{m}{2}\sum_{i=1}^{n}\dot{r}_i^2 - \frac{m^2}{2M}\left(\sum_{i=1}^{n}\dot{r}_i\right)^2 - \frac{1}{2}\sum_{ij}\frac{e_ie_j}{r_{ij}}$.

44. $T = \frac{M}{2}\dot{R}^2 + \sum_{j=1}^{n-1}\frac{\mu_j}{2}\dot{\rho}_j^2$, where $M = \sum_{j=1}^{n} m_j$ and $\frac{1}{\mu_j} = \frac{1}{\sum_{l=1}^{j} m_l} + \frac{1}{m_{j+1}}$.

45. $\mathscr{L} = \frac{M\dot{\mathbf{R}}^2}{2} + \frac{ml^2}{2}(\dot{\varphi}^2 + \dot{\theta}^2\sin^2\theta) - Ed\cos\theta$, where θ is the angle between the external electric field and the direction of the dipole moment, $d = el$, and $m = \frac{m_1m_2}{M}$.

46. $\mathbf{R} = \frac{mv_0^2\cos^2\alpha}{r}\mathbf{n}$, where $\mathbf{n}$ is a unit vector normal to the surface of the cylinder, and r the radius of the cylinder.

47. $x = r\cos\left(\frac{v_0\cos\alpha}{r}t\right)$

$$y = r\sin\left(\frac{v_0\cos\alpha}{r}t\right)$$

$$z = \frac{gt^2}{2} + v_0 t\sin\alpha.$$

48. $x = \left(a - \frac{g\cos\alpha}{\omega^2\sin^2\alpha}\right)\cosh(\omega t\sin\alpha) + \frac{g\cos\alpha}{\omega^2\sin^2\alpha}$.

49. $\varphi = \arc\cos\frac{3g}{2\omega^2 l}$, $\frac{3g}{2\omega^2 l} \ll 1$, $R = \frac{M\omega^2 l}{2}\sqrt{1+\frac{7g^2}{4l^2\omega^4}}$.

50. $l = \frac{a-b}{n+1}$.

51. $\Delta x = -\frac{h(m_1\cos\alpha + m_2\sin\alpha)}{m+m_1+m_2}$.

52. (1) $x = \frac{l(M+2m)}{M+M_1+m}\cos\omega t$;

(2) $R = (M+2m)\,l\omega^2$;

(3) $\omega \geqslant \sqrt{\frac{g}{l}\times\frac{M+M_1+m}{M+2m}}$.

53. The decay is possible if E_1+E_2 does not exceed E.

54. $x=0$, $y = \frac{g\omega\cos\psi}{3}t^3$, and $z = -\frac{gt^2}{2}$ if the time of fall satisfies the inequality $t\times 2\omega\sin\psi \ll 1$, where ψ is the latitude.

55. $\mathscr{L} = \frac{ml^2}{2}\dot{\varphi}^2 + mgl\cos\varphi - m\ddot{x}(t)\,l\cos\varphi - m\ddot{y}(t)\,l\sin\varphi$.

56. $\ddot{\varphi} + \omega_0^2(1+\alpha\cos\omega t)\sin\varphi = 0$, where $\omega_0^2 = \frac{g}{l}$ and $\alpha = \frac{a\omega^2}{g}$; for $\varphi \ll 1$, $\ddot{\varphi} + \omega_0^2(1+\alpha\cos\omega t)\varphi = 0$.

57. $\mathscr{L} = m_1a^2(\dot{\theta}^2 + \omega^2\sin^2\theta) + 2m_2a^2\dot{\theta}^2\sin^2\theta + 2(m_1+ + m_2)ga\cos\theta$, where $0<\theta<\pi/2$; the positions of equilibrium are

$$\theta_0 = 0 \text{ and } \cos\theta_0 = \frac{g}{a\omega^2}\,\frac{m_1+m_2}{m_1} \leqslant 1.$$

58. If $\cos\theta_0 = \frac{g}{a\omega^2}\,\frac{m_1+m_2}{m_1} > 1$, the system is in stable equilibrium when $\theta_0 = 0$. If $\cos\theta_0 < 1$, the system is in stable equilibrium when $\cos\theta_0 = \frac{g}{a\omega^2}\,\frac{m_1+m_2}{m_1}$.

59. $\omega_1 = \sqrt{\omega_0^2 + \frac{ca^2}{ml^2}}$, $\omega_2 = \sqrt{\omega_0^2 - \frac{ca^2}{ml^2}}$,

where $\omega_0^2 = \frac{g}{l} + \frac{ca^2}{ml^2}$;

$$\varphi_1 = \frac{\varphi_0}{2}(\cos\omega_1 t + \cos\omega_2 t), \quad \varphi_2 = \frac{\varphi_0}{2}(\cos\omega_1 t - \cos\omega_2 t).$$

60. *Hint.* The Kirchhoff law for an electric circuit with an e.m.f. states that

$$L\frac{d\dot{q}}{dt}+R\dot{q}+\frac{q}{C}+\mathscr{E}=0,\quad \dot{q}=I \tag{1}$$

where q is the capacitor's charge.

Since Eq. (1) coincides in form with the equation for a mechanical oscillator that is under the action of a constant force and also damped by friction, we can write the Lagrangian in the following form:

$$\mathscr{L}=\frac{L\dot{q}^2}{2}-\frac{q^2}{2C(x)}-\mathscr{E}q+\frac{m\dot{x}^2}{2}-\frac{c(x-l)^2}{2} \tag{2}$$

where l is the length of the spring in the unstretched position. We can then obtain an equation of type (1) through the Lagrangian (2) if we use the Lagrange equation of the second kind in the presence of dissipative forces $Q'=-R\dot{q}$.

61. $$\omega_{1,2}=\left\{\frac{(C_0L)^{-1}+c/m}{2}\pm\frac{1}{2}\left[\left(\frac{1}{C_0L}+\frac{c}{m}\right)^2+\frac{4q_0^2}{mC_0^2l^2L}\right]^{1/2}\right\}^{1/2}.$$

The state of equilibrium $\frac{\partial V}{\partial q}=0$ and $\frac{\partial V}{\partial x}=0$ can be found if we use Eq. (2) from the solution of Problem 60 and set $V=\frac{q^2}{2C(x)}+\mathscr{E}q+\frac{c(x-l)^2}{2}$.

62. $\omega=\sqrt{\frac{2ca^2}{ml^2}+\frac{g}{l}}$.

63. $\omega=\sqrt{\frac{2ca^2}{ml^2}-\frac{g}{l}}$.

The ball is in a state of stable equilibrium if $\frac{2ca^2}{ml^2}>\frac{g}{l}$.

64. The frequencies can be found from the equation

$$\omega^4-\left[\frac{c}{M}+\frac{g}{l}\,\frac{M+m}{M}\right]\omega^2+\frac{c}{M}\,\frac{g}{l}=0.$$

65. In an inertial frame of reference the position of mass m is

$$x = x(t) + l \sin \varphi, \qquad y = l \cos \varphi$$

and the Lagrangian is

$$\mathcal{L} = \frac{ml^2}{2} \dot{\varphi}^2 + mgl \cos \varphi - m\ddot{x} l \sin \varphi.$$

66. $\ddot{\varphi} + \omega_0^2 \varphi = \alpha \cos \gamma t$, where $\omega_0^2 = \frac{g}{l}$, $\alpha = \frac{a\gamma^2}{l}$.

67. $R_C = \frac{Pl}{4h} \sin 2\alpha$, $R_B = P\left(1 - \frac{l}{4h} \sin \alpha \sin 2\alpha\right)$, $T = \frac{Pl}{2h} \cos^2 \alpha \sin \alpha$.

68. $R_B = P$, $T_B = \frac{P}{2} \cot \alpha$, $R_A = \frac{P}{2} \sin \beta \cot \alpha$, $T_A = \frac{P}{2} \cos \beta \cot \alpha$.

69. $T = mg \frac{l}{d+r}$, $N = mg \frac{r}{d+r}$.

70. $\omega^2 = \frac{3g}{2(b^3 - a^3)} \left(\frac{b^2}{\sin \varphi} - \frac{a^2}{\cos \varphi} \right)$.

71. $\frac{b+d}{b} = \frac{l}{c}$, $P_{\mathrm{W}} = \frac{b}{a} P$.

72. $F = \frac{xl_1}{el} P$.

73. $\tan \varphi = \frac{a}{b} \frac{P_2 - P_1}{P_2 + P_1 + p(4a+b)}$.

74. The state of equilibrium can be found from the condition $\tan \varphi = 3$; stable equilibrium is obtained when $\cos \varphi > 0$.

75. $\{L_x, p_y\} = -p_z$, $\{L_y, p_z\} = -p_x$, $\{L_z, p_x\} = -p_y$,
$\{L_x, p_x\} = 0$, $\{L_y, p_y\} = 0$, $\{L_z, p_z\} = 0$,
$\{L_x, p_z\} = p_y$, $\{L_y, p_x\} = p_z$, $\{L_z, p_y\} = p_x$,
$\{L_x, \mathbf{p}\} = [\mathbf{i} \times \mathbf{p}]$, $\{L_y, \mathbf{p}\} = [\mathbf{j} \times \mathbf{p}]$,
$\{L_z, \mathbf{p}\} = [\mathbf{k} \times \mathbf{p}]$.

76. $\{L_x, L_y\} = -L_z$, $\{L_y, L_z\} = -L_x$, $\{L_z, L_x\} = -L_y$.

77. *Hint.* Since φ is a scalar function, it can depend only on the scalar combinations of $\mathbf{r}$ and $\mathbf{p}$, i.e. r^2, p^2, and $(\mathbf{p}\cdot\mathbf{r})$. Thus

$$\operatorname{grad} \varphi = \frac{\partial \varphi}{\partial r^2}\, 2\mathbf{r} + \frac{\partial \varphi}{\partial (\mathbf{p}\cdot\mathbf{r})}\, \mathbf{p}$$

$$\operatorname{grad} \varphi = \frac{\partial \varphi}{\partial p^2}\, 2\mathbf{p} + \frac{\partial \varphi}{\partial (\mathbf{p}\cdot\mathbf{r})}\, \mathbf{r}$$

The required relation can then be proved by straightforward calculations.

78. $ad - bc = 1$.

79. *Hint.* A vector $\mathbf{f}\,(\mathbf{r}, \mathbf{p})$ can be written in the form

$$\mathbf{f} = \mathbf{r}\varphi_1 + \mathbf{p}\varphi_2 + [\mathbf{r} \times \mathbf{p}]\, \varphi_3$$

where φ_1, φ_2, and φ_3 are scalar functions. The required relation can then be proved by straightforward calculations.

80. If $\mathbf{B}$ does not depend on coordinates, the vector potential can be written as $\mathbf{A} = \frac{1}{2}\,[\mathbf{r} \times \mathbf{B}]$; so

$$\mathcal{L} = \frac{\dot{\mathbf{r}}^2}{2} - \frac{\omega^2 \mathbf{r}^2}{2} + \frac{e}{2m}\, \dot{\mathbf{r}}\, [\mathbf{r} \times \mathbf{B}],$$

$$\mathcal{H} = \frac{1}{2}\left(\mathbf{p} - \frac{e}{2m}\,[\mathbf{r} \times \mathbf{B}]\right)^2 + \frac{\omega^2 \mathbf{r}^2}{2}.$$

81. $\mathcal{L} = \dfrac{M\dot{\mathbf{R}}^2}{2} + \dfrac{\mu \dot{\mathbf{r}}^2}{2} - \dfrac{\alpha}{r}$ and $\mathcal{H} = \dfrac{\mathbf{P}^2}{2M} + \dfrac{\mathbf{p}^2}{2\mu} + \dfrac{\alpha}{r}$,

where $M = m_1 + m_2$ is the total mass and $\mu = \dfrac{m_1 m_2}{M}$ the reduced mass.

82. $$\mathcal{L} = \sum_{i=1}^{n} \frac{m_i \dot{\mathbf{r}}_i^2}{2} - \frac{1}{8\pi\varepsilon_0} \sum_{ij=1}^{n} \frac{e_i e_j}{r_{ij}} - \sum_{i=1}^{n} e_i \varphi\,(\mathbf{r}_i,\ t) + \sum_{i=1}^{n} e_i \dot{\mathbf{r}}_i \mathbf{A}\,(\mathbf{r}_i,\ t),$$

$$\mathcal{H} = \sum_{i=1}^{n} \frac{[\mathbf{p}_i - e_i \mathbf{A}\,(\mathbf{r}_i,\ t)]^2}{2m_i} + \frac{1}{8\pi\varepsilon_0} \sum_{i \neq j}^{n} \frac{e_i e_j}{r_{ij}} + \sum_{i=1}^{n} e_i \varphi\,(\mathbf{r}_i,\ t).$$

83. *Hint*. Represent the vector potential in the form $\mathbf{A} = [\mathbf{H} \times \mathbf{r}]/2$, turn to the coordinates of the centre of mass and the separation coordinates, and add the total derivative of the function $\frac{eH}{2c}[\mathbf{r} \times \mathbf{R}]$. We then get

$$\mathscr{L} = \frac{M\dot{\mathbf{R}}^2}{2} + \frac{\mu\dot{\mathbf{r}}^2}{2} + \frac{e^2}{r} + \frac{e(m_2 - m_1)}{2Mc}(\mathbf{H}\cdot[\mathbf{r}\times\dot{\mathbf{r}}]) + \frac{e}{c}(\mathbf{H}\cdot[\mathbf{r}\times\dot{\mathbf{R}}]).$$

84. $\mathscr{H} = \frac{1}{2I}\left[p_\theta^2 + \frac{(p_\psi - p_\varphi\cos\theta)^2}{\sin^2\theta}\right] + \frac{p_\varphi^2}{2I_3} + Mgl\ \cos\theta.$

85. $[I + \mu(x^2 + y^2)]\dot{\varphi} + \mu(x\dot{y} - y\dot{x}) = \mu(x_0\dot{y}_0 - y_0\dot{x}_0)$, where I is the moment of inertia of the disc.

86. $\varphi = \frac{R\alpha\mu}{2(I+\mu R^2)}t^2$, $\xi = -\frac{MR}{M}\cos\left[\frac{\alpha}{2R} + \frac{R\alpha\mu}{2(I+\mu R^2)}\right]t^2$, and $\eta = -\frac{MR}{M}\sin\left[\frac{\alpha}{2R} + \frac{R\alpha\mu}{2(I+\mu R^2)}\right]t^2$, where φ is the angular displacement of the disc and ξ and η are the Cartesian coordinates of the disc's centre of mass relative to the centre of mass of the system.

87. (1) $I_1 = I_2 = I_3 = \frac{3}{8}m_{\mathrm{H}}a^2$;

(2) $I_1 = \frac{2m_{\mathrm{H}}m_0}{M}a^2\cos^2\alpha$, $I_2 = 2m_{\mathrm{H}}a^2\sin^2\alpha$, $I_3 = I_1 + I_2$;

(3) $I_1 = I_2 = \frac{3m_{\mathrm{N}}m_{\mathrm{H}}}{M}\left(b^2 - \frac{a^2}{3}\right) + \frac{m_{\mathrm{H}}a^2}{2}$, $I_3 = m_{\mathrm{H}}a^2$,

where M is the total mass of the molecule.

88. The inertia tensor will have the form

$$I_{ij} = \begin{vmatrix} \mu a^2 & 0 & 0 \\ 0 & \mu a^2 & 0 \\ 0 & 0 & 0 \end{vmatrix}$$

in the coordinate system whose z-axis passes through both atoms.

89. $a = 0.8$ Å.

90. The moments of inertia are related as $\frac{1}{2}:\frac{2}{3}:1$, and the frequencies of vibrations as $\sqrt{2} : \sqrt{3/2} : 1$.

91. (1) $I_{zz} = \frac{M}{12}(b^2+c^2)$, $I_{xx} = \frac{M}{12}(a^2+c^2)$, $I_{yy} = \frac{M}{12}(a^2+b^2)$;

(2) $I_{xx} = I_{yy} = I_{zz} = \frac{2M}{5} R^2$;

(3) $I_{xx} = I_{yy} = \frac{3M}{20}\left(R^2+\frac{h^2}{4}\right)$, $I_{zz} = \frac{3M}{10} R^2$;

(4) $I_{xx} = \frac{M}{5}(b^2+c^2)$, $I_{yy} = \frac{M}{5}(a^2+c^2)$, $I_{zz} = \frac{M}{5}(a^2+b^2)$;

(5) $I_{xx} = I_{yy} = I_{zz} = \frac{2M}{5} \times \frac{D^5-d^5}{D^3-d^3}$;

(6) $I_{xx} = I_{yy} = \frac{M}{2}\left(R^2+\frac{5}{4}r^2\right)$;

(7) $I_{zz} = \frac{M}{2}(R^2+r^2)$, $I_{xx} = I_{yy} = \frac{1}{12} l^2 + \frac{1}{4}(R^2+r^2)$;

(8) $I_{zz} = \frac{M}{12} Ma^2$, $I_{xx} = I_{yy} = \frac{M}{12} l^2 + \frac{M}{24} a^2$;

(9) $I_{zz} = \frac{5}{12} Ma^2$, $I_{xx} = I_{yy} = \frac{M}{12} l^2 + \frac{5M}{24} a^2$.

92. $I_3 \frac{d\omega_3}{dt} = N$; $\omega_3 = \frac{N}{I_3} t + \omega_0$,

$$\omega_2 = A \cos \omega \left[\left(t+\frac{I_3\omega_0}{N}\right)^2 + \varphi\right]$$

$$\omega_1 = A \sin \omega \left[\left(t+\frac{I_3\omega_0}{N}\right)^2 + \varphi\right]$$

where $\omega = \frac{(I_3-I)N}{2II_3}$ and ω_0, φ, A are constants of integration.

93. In securing the body in such a way the angle θ equals $\pi/2$. Then, by the kinematic equations of Euler (I-24), we have

$$\omega_1 = \dot{\psi} \sin \varphi, \quad \omega_2 = \dot{\psi} \cos \varphi, \quad \omega_3 = \dot{\varphi}$$

In our case we can write formula (I-31) for the kinetic energy of rotation:

$$K=\frac{1}{2}(I_1\sin^2\varphi+I_2\cos^2\varphi)\,\dot{\psi}^2+\frac{1}{2}I_3\dot{\varphi}^2 \tag{1}$$

From the fact that ψ is a cyclic (ignorable) coordinate it follows that

$$\frac{\partial\mathcal{L}}{\partial\dot{\psi}}=\frac{\partial K}{\partial\dot{\psi}}=(I_1\sin^2\varphi+I_2\cos^2\varphi)\,\dot{\psi}=c=\text{constant} \tag{2}$$

Since K is constant, we find $\dot{\varphi}$ and $\dot{\psi}$ from (1) and (2) as functions of angle φ, the parameters of the system, and the two integrals of motion K and c.

94. $K=\frac{1}{2}I\dot{\psi}^2\sin^2\alpha+\frac{1}{2}I_3(\dot{\varphi}+\dot{\psi}\cos\alpha)^2.$

95. $\mathcal{L}=\frac{3}{4}M(R-a)^2\dot{\varphi}^2-Mg(R-a)\cos\varphi$. The motion is the same as that of a simple pendulum of length $l=\frac{3}{2}(R-a)$.

96. Since $\dot{\alpha}=0$ and $\varphi=0$, it follows from the kinematic equations of Euler (I-24) that

$$\begin{aligned} \omega_x&=0, & \omega_1&=0\\ \omega_y&=0, & \omega_2&=\dot{\psi}\sin\alpha\\ \omega_z&=\dot{\psi}=\omega, & \omega_3&=\dot{\psi}\cos\alpha \end{aligned}$$

The dynamic equations of Euler (I-32) yield

$$N'_x=I\frac{d\omega_1}{dt}+(I_3-I)\,\omega_2\omega_3=(I_3-I)\frac{\omega^2}{2}\sin 2\alpha$$

Using formula (I-21) for the rotation matrix, we get

$$N_x=L'_x\cos\psi,\quad N_y=L'_y\sin\psi$$

Since the moment of forces is the sum of the moments of the equal forces of inertia that are applied to points O and B, we have

$$N_x=-hF_y,\quad N_y=-hF_x$$

Whence

$$F_x = -\frac{M\omega^2}{2h}\sin 2\alpha\left(\frac{l^2}{12}-\frac{R^2}{4}\right)\cos\omega t$$

$$F_y = \frac{M\omega^2}{2h}\sin\alpha\left(\frac{l^2}{12}-\frac{R^2}{4}\right)\sin\omega t$$

if for the moments of inertia we use the answer to Problem 91.

97. $I\dfrac{d\omega_1}{dt}+(I_3-I)\,\omega_2\omega_3 = mgl\sin\theta\cos\varphi$

$$I\frac{d\omega_2}{dt}+(I-I_3)\,\omega_1\omega_3 = -mgl\sin\theta\sin\varphi$$

$$I_3\frac{d\omega_3}{dt}=0$$

where l is the distance from the point of support to the centre of mass.

98. Using the answer to Problem 97 (we set $l = 0$), we get

$$I\frac{d\omega_1}{dt}+(I_3-I)\,\omega_2\omega_0=0,\quad \omega_3=\omega_0=\text{constant}$$

$$I\frac{d\omega_2}{dt}+(I-I_3)\,\omega_1\omega_0=0$$

whence

$$\omega_1 = A\cos(\omega t+\alpha)$$

$$\omega_2 = A\sin(\omega t+\alpha),\qquad \text{where}\qquad \omega=\frac{I_3-I}{I}\,\omega_0$$

The angular momentum **L** is constant since there are no external forces. To find the Euler angles as functions of time it is convenient to use a system of coordinates whose z-axis coincides with vector **L**. Using formula (I-21), we get

$$L\sin\theta\sin\varphi = L_1 = I\omega_1 = IA\cos(\omega t+\alpha)$$

$$L\sin\theta\cos\varphi = L_2 = I\omega_2 = IA\sin(\omega t+\alpha)$$

$$L\cos\theta = L_3 = I_3\omega_0$$

From the last equality it follows that

$$\theta = \arccos\frac{I_3\omega_0}{M}$$

From the first two equalities we find that

$$L^2 \sin^2\theta = I^2A^2, \quad A = \frac{1}{I}\sqrt{L^2 - I_3^2\omega_0^2}$$

$$\varphi = \omega t + \alpha, \quad \psi = (\omega_0 - \omega \cos\theta)\, t + \psi_0.$$

99. $x = \cos\psi \cos\varphi\, x' - \sin\varphi \cos\psi\, y' + \sin\psi\, z'$

$y = (\cos\theta \sin\varphi - \sin\theta \sin\psi \cos\varphi)\, x'$

$+ (\cos\theta \cos\varphi + \sin\theta \sin\psi \sin\varphi)\, y' + \sin\theta \cos\psi\, z'$

$z = (-\sin\theta \sin\varphi - \cos\theta \cos\psi \cos\varphi)\, x'$

$+ (-\sin\theta \cos\varphi + \cos\theta \sin\psi \sin\varphi)\, y' + \cos\theta \cos\psi\, z'.$

100. In the system of coordinates attached to the body

$$\omega_{x'} = \dot\psi \sin\varphi - \dot\theta \cos\psi \cos\varphi$$

$$\omega_{y'} = \dot\psi \cos\varphi + \dot\theta \cos\psi \sin\varphi$$

$$\omega_{z'} = \dot\varphi - \dot\theta \sin\psi.$$

101. The kinetic energy of the body of the coach is

$$K_1 = \frac{I'\dot\theta^2}{2}$$

where I' is the body's moment of inertia about an axis passing through point O perpendicular to the drawing; the kinetic energy of the flywheel

$$K_2 = M_2 h_1^2 \dot\theta^2 + \frac{I}{2}(\dot\psi^2 + \dot\theta^2 \cos^2\psi) + \frac{I_3}{2}(\dot\varphi - \dot\theta \sin\psi)^2$$

the kinetic energy of the counterbalance

$$K_3 = \frac{M_3}{3}(\dot x_3^2 + \dot y_3^2 + \dot z_3^2) = \frac{M_3}{2}[h_2^2\dot\psi^2 + (h_1 + h_2\cos\psi)^2\dot\theta^2]$$

where $x_3 = h_2 \sin\psi$, $y_3 = (h_1 + h_2\cos\psi)\sin\theta$, $z_3 = (h_1 + h_2\cos\psi)\cos\theta$ are the coordinates of the counterbalance's centre of mass in an inertial frame of reference with O as the origin.

The potential energy of the body in the earth's gravitational field is

$$V_1 = M_1 gl \cos\theta$$

The potential energy of the flywheel is

$$V_2 = M_2 g h_1 \cos\theta$$

The potential energy of the counterbalance is

$$V_3 = M_3 g (h_1 + h_2 \cos\psi) \cos\theta$$

If we construct the Lagrangian and then write the Lagrange equation, we see that $\varphi = \omega_0 t$ and $\theta = \psi = 0$ are its solution. To see whether this solution is stable we construct the Lagrangian of small deviations:

$$\mathscr{L} = \frac{1}{2} a_{11}\dot{\theta}^2 + \frac{1}{2} a_{22}\dot{\psi}^2 - b_{21}\dot{\theta}\psi + \frac{1}{2} c_{11}\theta^2 + \frac{1}{2} c_{22}\psi^2$$

where

$$a_{11} = I' + M_2 h_1^2 + I + M_3 (h_1 + h_2)^2$$

$$a_{22} = I + M_3 h_2^2, \quad b_{21} = I_3 \omega_0$$

$$c_{11} = g\,[M_1 l + M_2 h_1 + M_3 (h_1 + h_2)], \quad c_{22} = M_3 h_2 g$$

The solution $\varphi = \omega_0 t$, $\theta = \psi = 0$ will be stable if the following inequality holds:

$$I_2\omega_0 > \sqrt{c_{11}a_{22}} + \sqrt{c_{22}a_{11}}.$$

102. The centre of gravity is located at a distance l from point O:

$$l = \frac{4a}{3\pi} \times \frac{\rho_1 - \rho_2}{\rho_1 + \rho_2} = \sigma a$$

The state of equilibrium can be found from the equation

$$\cot^2\varphi = \frac{e^4 - \sigma^2}{\sigma^2 (1 - e^2)}$$

where e is the eccentricity of the ellipse. The equation has a solution if $e^2 \geqslant \sigma$. Besides, there is always one stable state of equilibrium $\varphi = \frac{\pi}{2}$ and one unstable state $\varphi = \frac{3\pi}{2}$.

103. In spherical coordinates (r, θ, φ)

$$u_{rr} = \frac{\partial u_r}{\partial r}, \quad u_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}$$

$$u_{\varphi\varphi} = \frac{1}{r\sin\theta}\frac{\partial u_\varphi}{\partial \varphi} + \frac{u_\theta}{r}\cot\theta + \frac{u_r}{r}$$

$$2u_{\theta\varphi}=\frac{1}{r}\left(\frac{\partial u_\varphi}{\partial\theta}-u_\varphi\cot\theta\right)+\frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial\varphi}$$

$$2u_{r\theta}=\frac{\partial u_\theta}{\partial r}-\frac{u_\theta}{r}+\frac{1}{r}\frac{\partial u_r}{\partial\theta},\quad 2u_{\varphi r}=\frac{1}{r\sin\theta}\frac{\partial u_r}{\partial\varphi}+\frac{\partial u_\varphi}{\partial r}-\frac{u_\varphi}{r}$$

In cylindrical coordinates $(r,\ \varphi,\ z)$

$$u_{rr}=\frac{\partial u_r}{\partial r},\qquad u_{\varphi\varphi}=\frac{1}{r}\frac{\partial u_\varphi}{\partial\varphi}+\frac{u_r}{r}$$

$$u_{zz}=\frac{\partial u_z}{\partial z},\qquad 2u_{\varphi z}=\frac{1}{r}\frac{\partial u_z}{\partial\varphi}+\frac{\partial u_\varphi}{\partial z}$$

$$2u_{rz}=\frac{\partial u_r}{\partial z}+\frac{\partial u_z}{\partial r},\qquad 2u_{r\varphi}=\frac{\partial u_\varphi}{\partial r}-\frac{u_\varphi}{r}+\frac{1}{r}\frac{\partial u_r}{\partial\varphi}.$$

104. Into Eq. (I-50) we must introduce an additional term, the centrifugal force of inertia $\rho\omega^2\mathbf{r}$. This force will obviously cause the displacement of the particles only in the radial direction, i.e. along r. Projecting the equation onto the unit vector $\mathbf{e}_r$ (see formulas (3-4), (3-18), (3-25) and (3-27) of Appendix 3), we obtain the equation

$$\frac{E(1-\sigma)}{(1+\sigma)(1-2\sigma)}\frac{d}{dr}\left[\frac{1}{r}(ru_r)\right]=-\rho\omega^2 r$$

which has a solution that satisfies the condition $\sigma_{rr}=0$ for $r=R$:

$$u_r=\frac{\rho\omega^2(1+\sigma)(1-2\sigma)}{8E(1-\sigma)}r\left[(3-2\sigma)R^2-r^2\right]$$

where E is the Young modulus and σ is the Poisson ratio; both are related to the elastic constants by the equalities

$$E=\frac{\mu(3\lambda+2\mu)}{\lambda+\mu},\quad \sigma=\frac{\lambda}{2(\lambda+\mu)}.$$

105. One transverse wave with a phase velocity of $x=\sqrt{\frac{c_{44}}{\rho}}$ and two transverse-longitudinal waves whose phase velocities satisfy the equation

$$x^4-\frac{1}{\rho}(c_{11}+c_{44})x^2+\frac{1}{\rho^2}\left[c_{11}c_{44}-\frac{(c_{11}+c_{12})}{4}(c_{12}-c_{11}+2c_{44})\sin^2 2\theta\right]=0$$

where c_{11}, c_{12}, and c_{44} are the elastic constants of the cubic lattice and θ is the angle between the wave vector and the 4-fold axis of symmetry. For $\theta = 0$ or $\theta = \frac{\pi}{2}$ the waves propagate parallel to the faces of the cube.

106. $\theta_l = \theta_0$, $\sin\theta_t = \frac{c_t}{c_l}\sin\theta_0$,

$$R_l = \left| \frac{c_t^2 \sin 2\theta_t \sin 2\theta_0 - c_l^2 \cos^2 2\theta_t}{c_t^2 \sin 2\theta_t \sin 2\theta_0 + c_l^2 \cos^2 2\theta_t} \right|^2$$

$$R_t = \left| \frac{2c_l c_t \sin 2\theta_0 \cos 2\theta_t}{c_t^2 \sin 2\theta_t \sin 2\theta_0 + c_l^2 \cos^2 2\theta_t} \right|^2 \times \frac{c_t \cos\theta_t}{c_l \cos\theta_0}$$

where θ_l is the reflection angle of the longitudinal wave, θ_t is the reflection angle of the transverse wave, c_l and c_t are the respective velocities, and $R_l + R_t = 1$.

Hint. The boundary conditions are of the form (I-51).

107. $\theta_l = \theta_0$, $\sin\theta_l = \frac{c_l}{c_t}\sin\theta_0$,

$$R_t = \left| \frac{c_t^2 \sin 2\theta_l \sin 2\theta_0 - c_l^2 \cos^2 2\theta_0}{c_t^2 \sin 2\theta_l \sin 2\theta_0 + c_l^2 \cos^2 2\theta_0} \right|^2$$

$$R_l = \left| \frac{2c_t c_l \sin 4\theta_0}{c_t^2 \sin 2\theta_l \sin 2\theta_0 + c_l^2 \cos^2 2\theta_0} \right|^2 \times \frac{c_l \cos\theta_l}{c_t \cos\theta_0}$$

where, as in Problem 106, $R_l + R_t = 1$.

108. $\frac{dp}{dy} = -\frac{\Delta p}{l} = \text{constant}$ and $v_y = -\frac{\Delta p}{2\eta l}\left(z^2 - \frac{d^2}{4}\right)$, where $\Delta p/l$ is the pressure drop per unit length.

109. (1) $j = \left(\frac{\Delta p}{l}\right)^2 \frac{d^3}{24\eta}$, $T = T_0$

$$+ \left(\frac{\Delta p}{l}\right)^2 \frac{1}{12\eta\varkappa}\left(\frac{d^4}{16} - z^4\right);$$

(2) $j = \left(\frac{\Delta p}{l}\right)^2 \frac{d^3}{6\eta}$, $T = T_0 + \left(\frac{\Delta p}{l}\right)^2 \frac{5d^4}{96\varkappa\eta}$

$$- \left(\frac{\Delta p}{l}\right)^2 \frac{d^3}{12\varkappa\eta} z - \left(\frac{\Delta p}{l}\right)^2 \frac{1}{6\eta\varkappa} z^4.$$

Hint. Show that in our case $d\varepsilon/dt=0$ and use Eq. (I-38).

110. (1) $\frac{dp}{dr}=\rho\frac{v_\varphi^2}{r}$; $v_\varphi=\frac{\omega_1 r_1^2}{r_1^2-r_2^2}\left(r-\frac{r_2^2}{r}\right)$;

(2) $v_\varphi=-\frac{\omega_2 r_2^2}{r_1^2-r_2^2}\left(r-\frac{r_1^2}{r}\right)$;

(3) $v_\varphi=\frac{\omega_1 r_1^2-\omega_2 r_2^2}{r_1^2-r_2^2}r+\frac{(\omega_2-\omega_1)r_1^2 r_2^2}{r_1^2-r_2^2}\cdot\frac{1}{r}$

and $v_z=v_r=0$.

111. $T=T_0+\frac{\eta}{\varkappa}\omega_2^2\frac{r_1^2 r_2^4}{(r_1^2-r_2^2)^2}\ln\frac{r_1}{r_2}\times\ln\frac{r^2}{r_1 r_2}$.

112. In cylindrical coordinates

$$\left(\frac{d\mathbf{v}}{dt}\right)_{\mathbf{e}_r}=-\frac{1}{\rho}\frac{\partial p}{\partial r}$$
$$+\nu\left[\frac{\partial^2 v_r}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 v_r}{\partial\varphi^2}+\frac{\partial^2 v_r}{\partial z^2}+\frac{1}{r}\frac{\partial v_r}{\partial r}-\frac{2}{r^2}\frac{\partial v_\varphi}{\partial\varphi}-\frac{v_r}{r^2}\right]$$

$$\left(\frac{d\mathbf{v}}{dt}\right)_{\mathbf{e}_\varphi}=-\frac{1}{\rho}\frac{1}{r}\frac{\partial p}{\partial\varphi}$$
$$+\nu\left[\frac{\partial^2 v_\varphi}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 v_\varphi}{\partial\varphi^2}+\frac{\partial^2 v_\varphi}{\partial z^2}+\frac{1}{r}\frac{\partial v_\varphi}{\partial r}+\frac{2}{r^2}\frac{\partial v_r}{\partial\varphi}-\frac{v_\varphi}{r^2}\right]$$

$$\left(\frac{d\mathbf{v}}{dt}\right)_{\mathbf{e}_z}=-\frac{1}{\rho}\frac{\partial p}{\partial z}+\nu\left[\frac{\partial^2 v_z}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 v_z}{\partial\varphi^2}+\frac{\partial^2 v_z}{\partial z^2}+\frac{1}{r}\frac{\partial v_z}{\partial r}\right]$$

In spherical coordinates

$$\left(\frac{d\mathbf{v}}{dt}\right)_{\mathbf{e}_r}=-\frac{1}{\rho}\frac{\partial p}{\partial r}+\nu\left[\frac{1}{r}\frac{\partial^2(rv_r)}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 v_r}{\partial\theta^2}\right.$$
$$+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_r}{\partial\varphi^2}+\frac{\cot\theta}{r^2}\frac{\partial v_r}{\partial\theta}-\frac{2}{r^2}\frac{\partial v_\theta}{\partial\theta}-\frac{2}{r^2\sin\theta}\frac{\partial v_\varphi}{\partial\varphi}$$
$$\left.-\frac{2v_r}{r^2}-\frac{2\operatorname{ctg}\theta}{r^2}v_\theta\right]$$

$$\left(\frac{d\mathbf{v}}{dt}\right)_{\mathbf{e}_\theta}=-\frac{1}{\rho}\frac{1}{r}\frac{\partial p}{\partial\theta}+\nu\left[\frac{1}{r}\frac{\partial^2(rv_\theta)}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial\theta^2}\right.$$
$$+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_\theta}{\partial\varphi^2}+\frac{\cot\theta}{r^2}\frac{\partial v_\theta}{\partial\theta}-\frac{1}{r^2}\frac{\cot\theta}{\sin\theta}\frac{\partial v_\varphi}{\partial\theta}$$
$$\left.+\frac{2}{r^2}\frac{\partial v_r}{\partial\theta}-\frac{v_\theta}{r^2\sin^2\theta}\right]$$

$$\left(\frac{d\mathbf{v}}{dt}\right)_{\mathbf{e}_\varphi} = -\frac{1}{\rho}\frac{1}{r\sin\theta}\frac{\partial p}{\partial\varphi} + \nu\left[\frac{1}{r}\frac{\partial^2(rv_\varphi)}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 v_\varphi}{\partial\theta^2}\right.$$
$$+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_\varphi}{\partial\varphi^2} + \frac{\cot\theta}{r^2}\frac{\partial v_\varphi}{\partial\theta} + \frac{2}{r^2\sin\theta}\frac{\partial v_r}{\partial\varphi}$$
$$\left.+\frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial v_\theta}{\partial\varphi} - \frac{v_\varphi}{r^2\sin^2\theta}\right].$$

113. The frequencies can be found from the equation

$$\tan\frac{\omega d}{c} = i\,\frac{c\rho_1}{c_l\rho} \tag{1}$$

where $\rho_1 c$ is the product of the liquid's density and the velocity of the waves in it, ρc_l is the same quantity for the plate. Since usually $c\rho_1$ is considerably smaller than $c_l\rho$, we can write Eq. (1) approximately in the form

$$\omega_n = \frac{n\pi c_l}{d} + i\,\frac{c\rho_1}{\rho d}, \quad \text{where } n = 1, 2, \ldots, \infty$$

These frequencies are complex quantities due to sound attenuation in solids. The physical meaning of this is that the energy of the sound wave is radiated into an unbounded medium (the liquid).

114. $\omega = xc_t k$, where x is a real root of the equation

$$\xi^6 - 8\xi^4 + 8\xi^2\left(3 - 2\frac{c_t^2}{c_l^2}\right) - 16\left(1 - \frac{c_t^2}{c_l^2}\right) = 0$$

The Rayleigh wave has two parts—the longitudinal part and the transverse part. The ratio between them is

$$\frac{u_z}{u_x} = -\frac{2-\xi}{2\sqrt{1-\xi^2}}.$$

115. Since the radial vibrations are longitudinal, curl $\mathbf{u} = 0$. Thus $\mathbf{u}$ can be represented as $\mathbf{u} = \operatorname{grad}\varphi$. In this case φ satisfies the equation $\ddot{\varphi} = c_l^2\Delta\varphi$. For radial monochromatic vibrations the equation takes the form:

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\varphi}{dr}\right) = -\frac{\omega^2}{c_l^2}\varphi$$

The solution of this equation is

$$\varphi = A\,\frac{\sin kr}{r}\,e^{i\omega t}$$

The boundary condition $\sigma_{rr}(R)=0$ yields

$$\frac{\tan\frac{\omega R}{c_l}}{\frac{\omega R}{c_l}}=\frac{1}{1-\left(\frac{\omega R}{2c_t}\right)^2}$$

The solutions of this equation are the natural frequencies of vibrations of the elastic sphere.

116. We seek the solution in the form of an outgoing spherical wave

$$\varphi=A\frac{e^{i\left(\omega t-\frac{\omega}{c_t}r\right)}}{r}$$

By the boundary condition we get

$$\omega=\frac{2ic_t^2}{c_l R}\pm\frac{2c_t}{R}\left(1-\frac{c_t^2}{c_l^2}\right)^{1/2}$$

The vibrations are damped since the energy dissipates in the form of sound waves.

117. $\omega^2=c^2\pi^2\left(\frac{n^2}{a^2}+\frac{m^2}{b^2}+\frac{l^2}{d^2}\right)$, where n, m, and l are integers that change from 1 to ∞, and c is the velocity of sound in the gas.

118. *Hint.* We must pass over to the frame of reference where the sound source is fixed. In this frame the field of velocities has the form $\mathbf{v}=\mathbf{v}_0+\mathbf{v}'$. We can now obtain the wave equation using the motion equation

$$\frac{\partial\mathbf{v}'}{\partial t}+(\mathbf{v}_0\cdot\nabla)\,\mathbf{v}'=-\frac{\operatorname{grad}p'}{\rho_0}$$

the equation of continuity

$$\frac{\partial\rho'}{\partial t}+(\mathbf{v}_0\cdot\operatorname{grad}\rho')+\rho_0\operatorname{div}\mathbf{v}'=0$$

and the thermodynamic equation $p=p(\rho)$, which relates the density and the pressure.

The dispersion equation has the form $(\omega-\mathbf{k}\mathbf{v}_0)^2=c^2k^2$ where $c^2=\left(\frac{dp}{d\rho}\right)_{\rho=\rho_0}$ is the square of the velocity of sound.

The frequency registered by the receiver is

$$\omega' = \omega\left(1 + \frac{v_0}{c}\cos\theta\right)^{-1}$$

where θ is the angle between the vectors $\mathbf{k}$ and $\mathbf{v}_0$, and ω is the frequency of the sound waves generated by the source.

119. $\omega' = \omega\left(1 - \frac{v_0}{c}\cos\theta\right)$. The notation is the same as in Problem 118.

120. $v_z = \frac{\Delta p}{4\eta l}\left[r_2^2 - r^2 + \frac{r_2^2 - r_1^2}{\ln\frac{r_2}{r_1}}\ln\frac{r}{r_2}\right]$.

121. $v_z = \frac{\Delta p}{2\eta l}\,\frac{a^2b^2}{a^2+b^2}\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)$,

$$Q = \rho\,\frac{\pi\Delta p}{4\eta l}\times\frac{a^3b^3}{a^2+b^2}$$

where a and b are the semiaxes of the ellipse.

122. $T = T_0 + \left(\frac{\Delta p}{l}\right)^2\frac{R^4}{64\varkappa\eta}\left[1 - \left(\frac{r}{R}\right)^4\right]$.

SECTION II

1. $\operatorname{grad} f(r) = \frac{df}{dr}\frac{\mathbf{r}}{r}$.

2. $\operatorname{div}\mathbf{r} = 3$, $\operatorname{curl}\mathbf{r} = 0$, $\operatorname{curl}\varphi(r)\,\mathbf{r} = 0$.

3. $\operatorname{grad}(\mathbf{P}\cdot\mathbf{r}) = \mathbf{P}$, $\operatorname{grad}\frac{(\mathbf{P}\cdot\mathbf{r})}{r^3} = \frac{\mathbf{P}}{r^3} - \frac{3(\mathbf{P}\cdot\mathbf{r})}{r^5}\mathbf{r}$, $(\mathbf{P}\cdot\nabla)\,\mathbf{r} = \mathbf{P}$, $\operatorname{div}[\mathbf{P}\times\mathbf{r}] = 0$, $\operatorname{curl}[\mathbf{r}\times\mathbf{P}] = -2\mathbf{P}$.

4. $\operatorname{grad}\mathbf{A}(r)\,\mathbf{B}(r) = \frac{\mathbf{r}}{r}(\dot{\mathbf{A}}\mathbf{B} + \dot{\mathbf{B}}\mathbf{A})$, $\operatorname{div}\varphi(r)\,\mathbf{A}(r) = \frac{\dot\varphi}{r}(\mathbf{r}\cdot\mathbf{A}) + \frac{\varphi}{r}(\mathbf{r}\cdot\dot{\mathbf{A}})$, $\operatorname{curl}\varphi(r)\,\mathbf{A}(r) = \frac{\dot\varphi}{r}[\mathbf{r}\times\mathbf{A}] + \frac{\varphi}{r}[\mathbf{r}\times\dot{\mathbf{A}}]$, where $\dot{\mathbf{A}} = \frac{d\mathbf{A}}{dr}$, $\dot{\mathbf{B}} = \frac{d\mathbf{B}}{dr}$, and $\dot\varphi = \frac{d\varphi}{dr}$.

5. We multiply the sought integral by a constant vector $\mathbf{p}$:

$$(\mathbf{p}\cdot\mathbf{I}) = \mathbf{p}\oint \mathbf{r}\,(\mathbf{A}\cdot\mathbf{n})\,dS = \oint (\mathbf{p}\cdot\mathbf{r})\,A_n\,dS$$
$$= \int \operatorname{div}(\mathbf{p}\cdot\mathbf{r})\,\mathbf{A}\,dV = \int \mathbf{A}\operatorname{grad}(\mathbf{p}\cdot\mathbf{r})\,dV$$
$$= \int (\mathbf{A}\cdot\mathbf{p})\,dV = (\mathbf{A}\cdot\mathbf{p})\,V$$

Since p is arbitrary,

$$\mathbf{I} = \mathbf{A}V$$

In a similar way we can show that

$$\oint (\mathbf{A}\cdot\mathbf{r})\,\mathbf{n}\,dS = \mathbf{A}V.$$

8. $\varphi = \frac{a}{r} + b.$

9. In solving this problem make use of Appendix 3. In cylindrical coordinates the Maxwell equations are:

$$\frac{1}{r}\frac{\partial H_z}{\partial \varphi} - \frac{\partial H_\varphi}{\partial z} = j_r + \frac{\partial D_r}{\partial t}$$
$$\frac{\partial H_r}{\partial z} - \frac{\partial H_z}{\partial r} = j_\varphi + \frac{\partial D_\varphi}{\partial t}$$
$$\frac{1}{r}\frac{\partial}{\partial r}(rH_\varphi) - \frac{1}{r}\frac{\partial H_r}{\partial \varphi} = j_z + \frac{\partial D_z}{\partial t}$$
$$\frac{1}{r}\frac{\partial E_z}{\partial \varphi} - \frac{\partial E_\varphi}{\partial z} = -\frac{\partial B_r}{\partial t}$$
$$\frac{\partial E_r}{\partial z} - \frac{\partial E_z}{\partial r} = -\frac{\partial B_\varphi}{\partial t}$$
$$\frac{1}{r}\frac{\partial}{\partial r}(rE_\varphi) - \frac{1}{r}\frac{\partial E_r}{\partial \varphi} = -\frac{\partial B_z}{\partial t}$$
$$\frac{1}{r}\frac{\partial}{\partial r}(rD_r) + \frac{1}{r}\frac{\partial D_\varphi}{\partial \varphi} + \frac{\partial D_z}{\partial z} = \rho$$
$$\frac{1}{r}\frac{\partial}{\partial r}(rB_r) + \frac{1}{r}\frac{\partial B_\varphi}{\partial \varphi} + \frac{\partial B_z}{\partial z} = 0$$

In spherical coordinates:

$$\frac{1}{r\sin\theta}\left\{\frac{\partial}{\partial\theta}(\sin\theta H_{\varphi})-\frac{\partial H_{\theta}}{\partial\varphi}\right\}=j_z+\frac{\partial D_r}{\partial t}$$

$$\frac{1}{r}\left\{\frac{1}{\sin\theta}\frac{\partial H_r}{\partial\varphi}-\frac{\partial}{\partial r}(rH_{\varphi})\right\}=j_{\theta}+\frac{\partial D_{\theta}}{\partial t}$$

$$\frac{1}{r}\left\{\frac{\partial}{\partial r}(rH_{\theta})-\frac{\partial H_r}{\partial\theta}\right\}=j_{\varphi}+\frac{\partial D_{\varphi}}{\partial t}$$

$$\frac{1}{r\sin\theta}\left\{\frac{\partial}{\partial\theta}(\sin\theta E_{\varphi})-\frac{\partial E_{\theta}}{\partial\varphi}\right\}=-\frac{\partial B_r}{\partial t}$$

$$\frac{1}{r}\left\{\frac{1}{\sin\theta}\frac{\partial E_r}{\partial\varphi}-\frac{\partial}{\partial r}(rE_{\varphi})\right\}=-\frac{\partial B_{\theta}}{\partial t}$$

$$\frac{1}{r}\left\{\frac{\partial}{\partial r}(rE_{\theta})-\frac{\partial E_r}{\partial\theta}\right\}=-\frac{\partial B_{\varphi}}{\partial t}$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2D_r)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta D_{\theta})+\frac{1}{r^2\sin^2\theta}\frac{\partial D_{\varphi}}{\partial\varphi}=\rho$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2B_r)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta B_{\theta})+\frac{1}{r^2\sin^2\theta}\frac{\partial B_{\varphi}}{\partial\varphi}=0.$$

10. $$\mathbf{E}=\begin{cases}\dfrac{1}{3\varepsilon_0}\rho\mathbf{r} & \text{for } r<R\\ \dfrac{1}{3\varepsilon_0}\rho\dfrac{R^3}{r^3}\mathbf{r} & \text{for } r>R.\end{cases}$$

11. From the superposition principle for fields it follows that the sought electric field is equal to the difference between the electric field intensity of a solid sphere and that of the charges that in this case are inside the cavity.

The electric field intensity inside the cavity is $\mathbf{E}=\frac{1}{3\varepsilon_0}\rho\mathbf{a}$;

inside the sphere but outside the cavity $\mathbf{E}=\frac{1}{3\varepsilon_0}\rho\mathbf{r}-\frac{1}{3\varepsilon_0}\rho\frac{R'^3}{|\mathbf{r}-\mathbf{a}|^3}(\mathbf{r}-\mathbf{a})$;

outside the sphere $\mathbf{E}=\frac{1}{3\varepsilon_0}\rho\left[\frac{R^3}{r^3}\mathbf{r}-\frac{R'^3}{|\mathbf{r}-\mathbf{a}|^3}(\mathbf{r}-\mathbf{a})\right]$,

where $\mathbf{a}$ is the radius vector connecting the centres of the sphere and cavity.

12. $$\mathbf{E}=\begin{cases}\dfrac{\alpha}{\varepsilon_0}\dfrac{r^n}{n+3}\mathbf{r} & \text{for } r<R\\ \dfrac{\alpha}{\varepsilon_0(n+3)}\dfrac{R^{n+3}}{r^3}\mathbf{r} & \text{for } r>R.\end{cases}$$

13. $$\mathbf{E}=\begin{cases}\dfrac{\varkappa}{2\pi\varepsilon_0 R^2}\mathbf{r} & \text{for } r<R\\ \dfrac{\varkappa}{2\pi\varepsilon_0 r^2}\mathbf{r} & \text{for } r>R\end{cases}$$

where r is the distance from the cylinder's axis to the observation point.

14. Let us direct the z-axis normally to the layer and select the origin of coordinates in the middle of the layer. Then

$$E=\frac{\rho z}{\varepsilon_0}\quad \text{inside the layer}$$

$$E=\frac{\rho d}{2\varepsilon_0}\frac{z}{|z|}\quad \text{outside the layer.}$$

15. (a) $C=\dfrac{4\pi\varepsilon R_1R_2}{R_2-R_1}$, where R_1 and R_2 are the radiuses of the capacitor plates ($R_2>R_1$);

(b) $C=\dfrac{\varepsilon S}{d}$, where S is the area of one capacitor plate, and d the distance between the plates;

(c) $C=\dfrac{4\pi\varepsilon l}{\ln\dfrac{R_2}{R_1}}$, where l is the length of the capacitor, and R_1 and R_2 are the radiuses of the plates.

16. Let us assume that the charge per unit length for the first conductor is $\varkappa$ and for the second $-\varkappa$. The potential of each conductor is composed of the potential φ_1 created by the conductor's charges and the potential φ_2 created by the charges of the other conductor. This second term can be considered to be the same at every point of the conductor provided the distance between the conductors is large. For the first conductor we then have

$$\varphi_1=-\frac{\varkappa}{2\pi\varepsilon_0}\ln R_1+\frac{\varkappa}{2\pi\varepsilon_0}\ln d$$

for the second

$$\varphi_2 = \frac{\varkappa}{2\pi\varepsilon_0}\ln R_2 - \frac{\varkappa}{2\pi\varepsilon_0}\ln d$$

The capacitance per unit length of this system is

$$C = \frac{\varkappa}{\varphi_1 - \varphi_2} = \frac{2\pi\varepsilon_0}{\ln \dfrac{d^2}{R_1 R_2}} = \pi\varepsilon_0 \left(\ln \frac{d}{R}\right)^{-1}$$

where $R = \sqrt{R_1 R_2}$ is the geometric mean of the radiuses of the conductors.

17. In cylindrical coordinates with the z-axis directed along the line that connects the two charges, the equation of the lines of force is

$$\frac{dr}{E_r} = \frac{r\,d\varphi}{E_\varphi} = \frac{dz}{E_z} \tag{1}$$

In our case

$$\begin{aligned} E_r &= \frac{er}{4\pi\varepsilon_0 [r^2 + (z - d/2)^2]^{3/2}} - \frac{er}{4\pi\varepsilon_0 [r^2 + (z + d/2)^2]^{3/2}} \\ E_\varphi &= 0 \\ E_z &= \frac{e(z - d/2)}{4\pi\varepsilon_0 [r^2 + (z - d/2)^2]^{3/2}} - \frac{e(z + d/2)}{4\pi\varepsilon_0 [r^2 + (z + d/2)^2]^{3/2}} \end{aligned} \tag{2}$$

Substitute the components of the electric field (2) into (1). Now go over to new variables u and v instead of r and z in the following manner:

$$z = \frac{d}{2}\,\frac{u+v}{u-v}, \quad r = \frac{d}{u-v} \tag{3}$$

We come to the equation

$$\frac{dv}{(1+v^2)^{3/2}} = \frac{du}{(1+u^2)^{3/2}} \tag{4}$$

Integrating Eq. (4), we obtain

$$\frac{v}{(1+v^2)^{1/2}} - \frac{u}{(1+u^2)^{1/2}} = C$$

or

$$\frac{z - d/2}{[r^2 + (z - d/2)^2]^{1/2}} - \frac{z + d/2}{[r^2 + (z + d/2)^2]^{1/2}} = C$$

The pattern of the lines of force is given in Fig. 51.

18. Let the x-axis be directed along the field. Then

$$W = -|\mathbf{E}|(x + iy) = \varphi + i\Psi,$$

where $\varphi = -|\mathbf{E}|x$ and $\Psi = -|\mathbf{E}|y$. Such a field is created by a uniformly charged surface $y = 0$, for instance. In this case $|\mathbf{E}| = \sigma/(2\varepsilon_0)$.

19. Let us use a conformal mapping $W = Az^2 i$ to transform the angle into a half-plane, for which we have the solution from Problem 18. The complex-valued potential is

$$W = \varphi + i\Psi = Az^2 i$$
$$= A(x + iy)^2 i$$

Whence

$$\varphi = -2Axy \quad \text{and}$$
$$\Psi = A(x^2 - y^2)$$

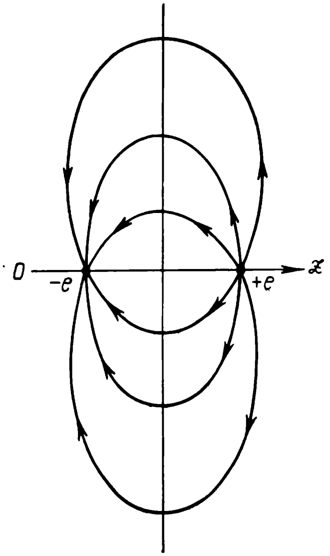

Fig. 51

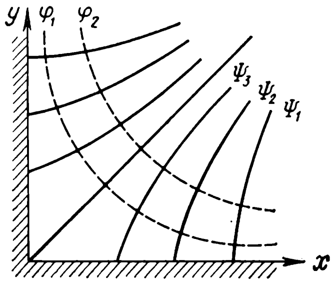

Fig. 52

The equipotential surfaces and the lines of force are depicted in Fig. 52.

20. The complex-valued potential $W = \varphi + i\Psi = \sqrt{z}$. Hence

$$x + iy = \varphi^2 - \Psi^2 + 2i\varphi\Psi$$
$$\varphi^2 - \Psi^2 = x, \quad 2\varphi\Psi = y$$

Excluding φ and Ψ, we obtain the following equations:

$$\frac{y^2}{4\varphi^2} = \varphi^2 - x, \quad \frac{y^2}{4\Psi^2} = x + \Psi^2$$

Let us find the lines φ = constant. If $\varphi = C$, we come to the parabola

$$y^2 = 4C^2 (C^2 - x)$$

which is turned in the negative direction of the x-axis and whose vertex is at point $x = C$. If $C = 0$, the parabola turns into a straight line (semiaxis)

$$y = 0, \quad x < 0$$

Thus the function $\varphi = \mathrm{Re}\sqrt{z}$ gives the potential near a grounded half-line $y = 0$, $x < 0$. The equipotential surfaces are a system of parabolas.

21. Writing z in the form $z = re^{i\theta}$, we find that $\varphi = \ln r$. The equipotential surfaces are a system of circles with radiuses r = constant. In the three-dimensional case such a potential is created by a charged straight line that lies along the z-axis.

22. Let us write the equation of the parabola in parametric form

$$x = ap^2 - a, \quad y = 2ap$$

where p changes from $-\infty$ to $+\infty$. According to Eq. (II-38),

$$z = aW^2 - a + 2aiW = (W + i)^2 a$$

Hence

$$W = \sqrt{z/a} - i$$

The potential is determined by the imaginary part of this function.

23. Writing the equation of the ellipse in parametric form $x = a\cos\theta$, $y = b\sin\theta$ and assuming that $a = C\cosh\alpha$, $b = C\sinh\alpha$ $(a^2 - b^2 = C^2)$, we obtain

$$z = a\cos W + ib\sin W = C\cos[\Psi + i(\varphi - \alpha)]$$

This suggests that

$$\begin{aligned} x &= C\cos\Psi\cosh(\varphi - \alpha) \\ y &= C\sin\Psi\sinh(\varphi - \alpha) \end{aligned} \tag{1}$$

Eliminating Ψ and setting φ = constant, we come to the equation of the equipotential surfaces:

$$\frac{x^2}{C^2\cosh^2(\varphi - \alpha)} + \frac{y^2}{C^2\sinh^2(\varphi - \alpha)} = 1$$

This is a system of ellipses whose focuses coincide with the focuses of the grounded ellipse.

Eliminating φ from (1) and setting Ψ = constant, we come to the equation of the lines of force

$$\frac{x^2}{C^2 \cos^2 \Psi} - \frac{y^2}{C^2 \sin^2 \Psi} = 1$$

This is a system of hyperbolas whose focal distances are the same as for the ellipses. When $a = b$,

$$z = a(\cos W + i \sin W) = ae^{i(\Psi + i\varphi)}$$

$$x = ae^{-\varphi} \cos \Psi$$

$$y = ae^{-\varphi} \sin \Psi$$

whence $e^{-\varphi} = \frac{x^2+y^2}{a^2}$, $\varphi = -2 \ln r/a$.

25. Using Eq. (II-39) for the potential of charges on a surface in cylindrical coordinates, we have

$$\varphi = \int_0^{2\pi} d\varphi \int_{R_1}^{R_2} \frac{\sigma r \, dr}{4\pi\varepsilon_0 \sqrt{r^2+z^2}} = \frac{\sigma}{2\varepsilon_0} \left(\sqrt{R_2^2+z^2} - \sqrt{R_1^2+z^2}\right)$$

where z is the coordinate of the observation point on the axis. The components of the electric field intensity vector are

$$E_x = E_y = 0, \quad E_z = \frac{\sigma}{2\varepsilon_0} \left(\frac{z}{\sqrt{R_1^2+z^2}} - \frac{z}{\sqrt{R_2^2+z^2}} \right)$$

For the limiting cases we have:

(a) $E_z = \frac{\sigma}{2\varepsilon_0} \left(\frac{z}{|z|} - \frac{z}{\sqrt{R_2^2+z^2}} \right)$;

(b) $E_z = \frac{\sigma}{2\varepsilon_0} \frac{z}{|z|}$.

26. For a system possessing spherical symmetry the potential is

$$\varphi(r) = \frac{1}{\varepsilon_0 r} \int_0^r \rho(r') r'^2 \, dr' + \frac{1}{\varepsilon_0} \int_r^\infty r' \rho(r') \, dr'$$

Substituting the known charge density, we obtain

$$\varphi = \frac{e}{4\pi\varepsilon_0 r}\left(1 - e^{-2r/a}\right) - \frac{e}{4\pi\varepsilon_0 a}\, e^{-2r/a}.$$

27. The potential of a point charge is given by the solution of the equation

$$\Delta\varphi = -\frac{e}{\varepsilon_0}\,\delta(\mathbf{r}) \tag{1}$$

Let us represent $\varphi(\mathbf{r})$ and $\delta(\mathbf{r})$ in the form of Fourier expansions:

$$\left.\begin{aligned} \varphi(\mathbf{r}) &= \int \varphi(\mathbf{k})\, e^{i\mathbf{kr}}\, d\mathbf{k} \\ \delta(\mathbf{r}) &= \frac{1}{(2\pi)^3}\int e^{i\mathbf{kr}}\, d\mathbf{k} \end{aligned}\right\} \tag{2}$$

Substituting integrals (2) into Eq. (1) and equating in the integrands the coefficients of $e^{i\mathbf{kr}}$, we obtain the expression for the Fourier transform of the potential:

$$\varphi(\mathbf{k}) = \frac{e}{\varepsilon_0 (2\pi)^3 k^2}.$$

28. $\varphi = \dfrac{\rho_0}{\varepsilon_0 (a^2+b^2+c^2)} \sin ax \sin by \sin cz.$

29. The solution which satisfies the Laplace equation and the boundary condition ($\varphi = 0$ on the conducting plane) is

$$\varphi = \frac{e}{4\pi\varepsilon_0 r} - \frac{e}{4\pi\varepsilon_0 r'}$$

$$\mathbf{E} = \frac{e\mathbf{r}}{4\pi\varepsilon_0 r^3} - \frac{e\mathbf{r}'}{4\pi\varepsilon_0 r'^3}$$

where $r = \sqrt{(x-d)^2+y^2+z^2}$ and $r' = \sqrt{(x+d)^2+y^2+z^2}$.

We assume that the grounded plane is plane $x = 0$. The radius vector $\mathbf{r}$ is drawn from the charge to the observation point, and $\mathbf{r}'$ from the charge's image, i.e. from point $x = -d$, $y = z = 0$, to the observation point.

The surface density of the induced charge

$$\sigma = -\frac{ed}{2\pi\sqrt{(y^2+z^2+d^2)^3}}$$

The total induced charge is

$$-\int_{-\infty}^{\infty}\int \frac{ed\,dx\,dy}{2\pi\sqrt{(y^2+z^2+d^2)^3}}=-e.$$

30. $\varphi=\frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r_1}-\frac{q}{r_2}+\frac{q}{r_3}-\frac{q}{r_4}\right)$, where r_1 is the the distance from the observation point P to the charge (Fig. 53).

31. The potential inside the sphere and on it is zero.

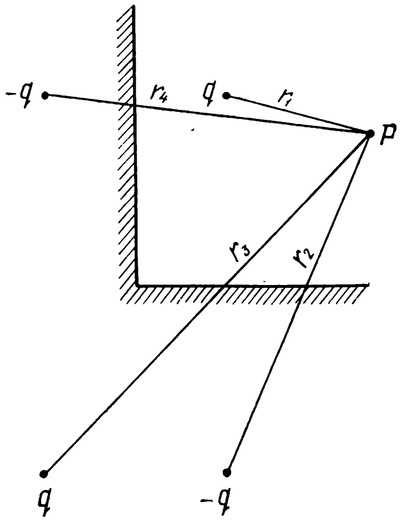

Fig. 53

Outside the sphere it must satisfy the equation

$$\Delta\varphi=-\frac{e}{\varepsilon_0}\delta(\mathbf{r})$$

The origin of coordinates coincides with the position of the point charge. The solution of this equation is sought in the form

$$\varphi=\frac{e}{4\pi\varepsilon_0 r}-\frac{e'}{4\pi\varepsilon_0 r'} \tag{1}$$

Vector $\mathbf{r}'$ is drawn from the observation point to a point inside the sphere. The position of the second point is determined from the boundary condition $\varphi = 0$ on the sphere.

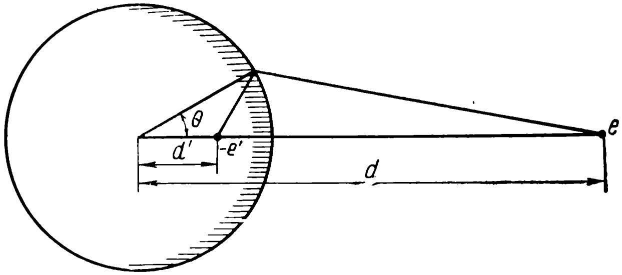

Fig. 54

The second member in the right-hand side of (1) may be thought of as the potential of a charge e' (the image) inside the sphere. Actually there is no such charge. But the real charge induced on the surface of the sphere acts in the same way as a certain charge would without the sphere. The symmetry of the problem implies that charge e' must lie on the line that connects the centre of the sphere with charge e placed at a distance d_1 from the centre. From the boundary conditions it follows that (Fig. 54)

$$\frac{e^2}{e'^2} = \frac{r^2}{r'^2} = \frac{R^2 + d^2 - 2dR\cos\theta}{R^2 + d_1^2 - 2d_1R\cos\theta}$$

This condition holds for any angle θ if

$$R^2 = dd_1, \quad \text{and} \quad e' = e\sqrt{d_1/d}$$

where d_1 determines the position of e'.

32. $\varphi = \frac{1}{4\pi\varepsilon_0}\left(\frac{e}{r} - \frac{e'}{r'} + \frac{e'}{r_0}\right)$, where r and r' have the same meaning as in Problem 31, and r_0 is the distance from the centre of the sphere to the observation point. If we consider only the first two members in the parentheses, the potential on the surface of the sphere is zero. The third member yields a constant value to the potential on the surface. The sphere remains neutral. The potential may be interpreted as having three terms: one from charge e,

another from charge $-e'$ in the conjugate point, and the third from e' in the centre of the sphere.

33. $\varphi = \frac{1}{4\pi\varepsilon_0}\left(\frac{e}{r_1} - \frac{e'}{r_2} + \frac{e'}{r_3} - \frac{e}{r_4}\right)$ (Fig. 55); here

$$e' = e\sqrt{d_1/d} \quad \text{and} \quad d_1 = R^2/d$$

In Fig. 55, P is the observation point.

34. Each charge of the dipole induces an image charge. Since the distances from the charges to the sphere are different, the magnitudes of the image charges will also be different. For this reason we must place a dipole $\mathbf{p}'$, oriented in the same way as $\mathbf{p}$, and a charge e' at a point that lies at a distance $d' = R^2/d$ from the sphere. We must also require that the potential

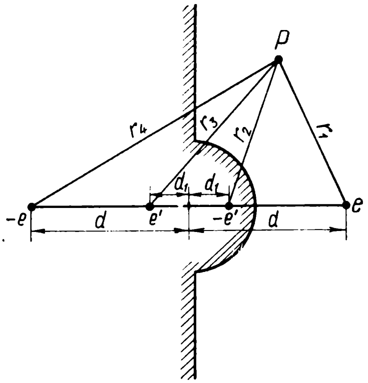

Fig. 55

$$\varphi = \frac{\mathbf{pr}}{4\pi\varepsilon_0 r^3} + \frac{\mathbf{p}'\mathbf{r}'}{4\pi\varepsilon_0 r'^3} + \frac{e'}{4\pi\varepsilon_0 r'} \quad (1)$$

of the system be zero on the surface of the sphere. We must remember (see Problem 31) that on the surface of the sphere

$$r = \sqrt{R^2 + d^2 - 2dR\cos\theta}$$
$$r' = \sqrt{R^2 + d'^2 - 2d'R\cos\theta}$$
$$\mathbf{pr} = p\,(d - R\cos\theta)$$
$$\mathbf{p}'\mathbf{r}' = p'\,(d' - R\cos\theta)$$

From the condition that $\varphi = 0$ when $r = R$ we find that

$$p\,(d - R\cos\theta) - \frac{d^3}{R^3}\,p'\left(R\cos\theta - \frac{R^2}{d}\right) + \frac{e'd}{R}\,(R^2 + d^2 - 2Rd\cos\theta) = 0$$

This condition must hold for any θ. Hence

$$pd + \frac{d^2}{R} p' + \frac{e'd}{R}(R^2 + d^2) = 0$$

$$pR + \frac{d^3}{R^2} p' + 2e'd^2 = 0$$

Solving these equations for e' and p' we have

$$e' = -\frac{R}{d^2} p \quad \text{and} \quad p' = \frac{R^3}{d^3} p$$

These relations together with (1) give the potential of the system.

35. In the medium with the charge the solution will be sought in the form

$$\varphi_1 = \frac{e}{4\pi\varepsilon_1\varepsilon_0 r} - \frac{e'}{4\pi\varepsilon_1\varepsilon_0 r'} \tag{1}$$

Vector $\mathbf{r}'$ is drawn from a point that is the mirror image of the coordinate of the charge in the interface of the two media.

In the other medium

$$\varphi_2 = \frac{e''}{4\pi\varepsilon_2\varepsilon_0 r} \tag{2}$$

The electric displacement vectors in the two media are determined by the relations

$$4\pi\mathbf{D}_1 = \frac{e\mathbf{r}}{r^3} - \frac{e'\mathbf{r}'}{r'^3}, \quad 4\pi\mathbf{D}_2 = \frac{e''\mathbf{r}}{r^3} \tag{3}$$

From the continuity of the potential ($\varphi_1 = \varphi_2$ when $r = r'$) and the normal components of the electric displacement vectors ($D_{n_1} = D_{n_2}$ when $r = r'$) we find that

$$\frac{e - e'}{\varepsilon_1} = \frac{e''}{\varepsilon_2} \quad \text{and} \quad e' + e = e''$$

Whence

$$e' = \frac{\varepsilon_2 - \varepsilon_1}{\varepsilon_1 + \varepsilon_2} e \quad \text{and} \quad e'' = \frac{2\varepsilon_2}{\varepsilon_1 + \varepsilon_2} e$$

By substituting e' and e'' into Eqs. (1)-(3) we can find the potential and the electric displacement vector in any point of space.

36. We seek the potential in the form

$$\varphi = \frac{a}{r}$$

which satisfies the boundary conditions $\varphi_1 = \varphi_2$ and $\varepsilon_1 \left(\frac{\partial \varphi}{\partial n}\right)_1 = \varepsilon_2 \left(\frac{\partial \varphi}{\partial n}\right)_2$ on the interface of the two dielectrics. The constant quantity a can be related to the charge of the sphere

$$e = \oint \sigma \, dS = -\int \varepsilon\varepsilon_0 \frac{\partial \varphi}{\partial n} \, dS$$

Here $\varepsilon = \varepsilon_1$ in medium *1* and $\varepsilon = \varepsilon_2$ in medium *2*.

After integrating the last expression we find that

$$a = \frac{e}{2\pi(\varepsilon_1 + \varepsilon_2)}$$

whence

$$4\pi \mathbf{D}_1 = \frac{2\varepsilon_1}{\varepsilon_1 + \varepsilon_2} \frac{e\mathbf{r}}{r^3}, \quad 4\pi \mathbf{D}_2 = \frac{2\varepsilon_2}{\varepsilon_1 + \varepsilon_2} \frac{e\mathbf{r}}{r^3}$$

$$\sigma_1 = \frac{\varepsilon_1 e}{2\pi(\varepsilon_1 + \varepsilon_2) R^2}, \quad \sigma_2 = \frac{\varepsilon_2 e}{2\pi(\varepsilon_1 + \varepsilon_2) R^2}.$$

37. Let us place the coordinate origin in the centre of the sphere and direct the polar axis along the straight line that connects the charge and the centre of the sphere. The equation

$$\Delta \varphi = -\frac{e}{\varepsilon_0} \delta(\mathbf{r} - \mathbf{d})$$

($\mathbf{d}$ is the radius vector of the charge) has a solution that vanishes at infinity:

$$\varphi(r, \theta) = \frac{e}{4\pi\varepsilon_0 |\mathbf{r} - \mathbf{d}|} + \sum_{l=0}^{\infty} \frac{b_l}{r^{l+1}} P_l(\cos\theta) \qquad (1)$$

where $P_l(\cos\theta)$ are Legendre polynomials, and b_l are numbers that can be found from the boundary conditions.

We now use the expansion

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \sum_{l=0}^{\infty} \frac{r^l}{(r')^{l+1}} P_l(\cos\theta) \quad \text{for} \quad r < r' \qquad (2)$$

where θ is the angle between $\mathbf{r}$ and $\mathbf{r}'$.

From the boundary condition ($\varphi = 0$ when $r = R$) it follows that

$$\sum_{l=0}^{\infty}\left(\frac{e}{4\pi\varepsilon_0}\frac{R^l}{d^{l+1}}+\frac{b_l}{R^{l+1}}\right)P_l(\cos\theta)=0$$

Since Legendre polynomials are orthogonal functions, the last expression is valid only if the coefficients of the polynomials are zero, or

$$b_l=-\frac{e}{4\pi\varepsilon_0}\frac{R^{2l+1}}{d^{l+1}}$$

$$\varphi=\frac{e}{4\pi\varepsilon_0\,|\mathbf{r}-\mathbf{d}|}-\frac{eR}{4\pi\varepsilon_0 d}\sum_{l=0}^{\infty}\left(\frac{R^2}{d}\right)^l\frac{P_l(\cos\theta)}{r^{l+1}} \tag{3}$$

Using the expansion (2), we can write the last member in the right-hand side of (3) as the potential of a point charge $e' = -eR/d$ placed on the line connecting this charge and the centre of the sphere, at a distance $d_1 = R^2/d$ from the centre. Thus

$$\varphi=\frac{e}{4\pi\varepsilon_0\,|\mathbf{r}-\mathbf{d}|}-\frac{|e'|}{4\pi\varepsilon_0\,|\mathbf{r}-\mathbf{d}_1|}$$

which coincides with the solution of Problem 31 (found by the method of images).

The charge density on the surface of the sphere

$$\sigma(R,\theta)=-\frac{e}{4\pi}\sum_{l=0}^{\infty}(2l+1)\frac{R^{l-1}}{d^{l+1}}P_l(\cos\theta)$$

The total charge induced on the sphere is $e' = -\dfrac{eR}{d}$.

38. $$\varphi=\frac{e}{4\pi\varepsilon_0\,|\mathbf{r}-\mathbf{d}|}+\frac{VR}{r}-\frac{eR}{4\pi\varepsilon_0 d}\sum_{l=0}^{\infty}\left(\frac{R^2}{d}\right)^l\frac{P_l(\cos\theta)}{r^{l+1}}$$

$$\sigma(R,\theta)=\frac{V}{4\pi R}-\frac{e}{4\pi}\sum_{l=0}^{\infty}(2l+1)\frac{R^{l-1}}{d^{l+1}}P_l(\cos\theta).$$

39. The potential may be represented as follows:

inside the sphere $\varphi_1 = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$,

outside the sphere $\varphi_2 = \sum_{l=0}^{\infty} B_l r^{-(l+1)} P_l(\cos\theta)$

On the sphere the boundary conditions (II-31) and (II-32) are

$$\varphi_1 = \varphi_2,$$

$$\frac{\partial\varphi_1}{\partial r} - \frac{\partial\varphi_2}{\partial r} = \frac{\sigma}{\varepsilon_0}$$

Substituting φ_1 and φ_2, we obtain

$$A_1 = \frac{\sigma_0}{3\varepsilon_0}, \quad B_1 = \frac{\sigma_0}{3\varepsilon_0} R^3,$$

$$A_l = B_l = 0 \quad \text{for} \quad l \neq 1$$

Thus

$$\varphi_1 = \frac{\sigma_0}{3\varepsilon_0} r\cos\theta, \quad \varphi_2 = \frac{\sigma_0}{3\varepsilon_0}\frac{R^3}{r^2}\cos\theta.$$

40. Knowing polarization P, we can find the densities of the bound surface (σ') and bound body (ρ') charges. For a uniformly polarized ball $\rho' = 0$ and $\sigma' = P\cos\theta$. According to the solution of Problem 39,

$$\varphi_1 = \frac{1}{3\varepsilon_0} Pr\cos\theta \quad \text{for} \quad r < R$$

$$\varphi_2 = \frac{1}{3\varepsilon_0} P \frac{R^3}{r^2}\cos\theta \quad \text{for} \quad r > R$$

$$\mathbf{E}_1 = -\frac{1}{3\varepsilon_0}\mathbf{P} \quad \text{for} \quad r < R$$

$$\mathbf{E}_2 = \frac{R^3(\mathbf{P}\cdot\mathbf{r})\,\mathbf{r}}{\varepsilon_0 r^5} - \frac{R^3}{3\varepsilon_0}\frac{\mathbf{P}}{r^3} \quad \text{for} \quad r > R.$$

41. When there is no sphere, the potential of an external field in a region without charges satisfies the Laplace equation, i.e.

$$\varphi_{\text{ext}} = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} A_{lm} r^l P_{lm}(\cos\theta)\, e^{im\alpha} \qquad (1)$$

where r, θ, α are the spherical coordinates of the observation point, and P_{lm} $(\cos\theta)$ are associated Legendre functions.

For a given field the A_{lm}' s are considered known. If we place a conducting sphere in the field, the potential of the sphere

$$\varphi = \varphi_{\text{sphere}} + \varphi_{\text{ext}} \tag{2}$$

where $\varphi_{\text{sphere}} = \sum\limits_{l, m} B_{lm} r^{-l-1} P_{lm} (\cos\theta)\, e^{im\alpha}$ is the potential of the charges induced on the sphere by the external field. We can determine the B_{lm}'s from the boundary conditions $\varphi = 0$ when $r = R$:

$$B_{lm} = -A_{lm} R^{2l+1} \tag{3}$$

Equations (1)-(3) fully determine the potential in the vicinity of the sphere.

42. Inside the sphere and on its surface $\varphi = 0$. Outside the sphere the potential can be sought in the form

$$\varphi = \sum_{l} b_l r^{-l-1} P_l (\cos\theta) - E_0 r P_1 (\cos\theta)$$

Imposing the boundary conditions on the potential, we get

$$b_1 = E_0 R^3$$

$$b_l = 0 \quad \text{for} \quad l \neq 1$$

Whence

$$\varphi = -E_0 r \cos\theta + \frac{E_0 R^3}{r^2} \cos\theta, \quad \sigma = 3\varepsilon_0 E_0 \cos\theta.$$

43. Inside the sphere

$$\varphi_1 = \sum_{l=0}^{\infty} a_l r^l P_l (\cos\theta);$$

outside the sphere

$$\varphi_2 = a_0' + a_1' r P_1 (\cos\theta) + \sum_{l=0}^{\infty} b_l r^{-l-1} P_l (\cos\theta)$$

The conditions of the problem imply that at large distances from the sphere

$$\varphi_2 \,|_{r\to\infty} \to -E_0 z = -E_0 r \cos\theta = -E_0 r P_1 (\cos\theta)$$

Whence

$$a_0' = 0, \quad a_1' = -E_0$$

Imposing the boundary conditions for the surface of the sphere ($r = R$), we get

$$\sum_{l=0}^{\infty} a_l R^l P_l(\cos\theta) = \sum_{l=0}^{\infty} b_l R^{-l-1} P_l(\cos\theta) - E_0 R P_1(\cos\theta)$$

$$\varepsilon \sum_{l=0}^{\infty} a_l l R^{l-1} P_l(\cos\theta) = -\sum_{l=0}^{\infty} b_l (l+1) R^{-l-2} P_l(\cos\theta) - E_0 P_1(\cos\theta)$$

From the two expressions it follows that

$$a_1 = -\frac{3E_0}{\varepsilon+2}, \quad b_1 = \frac{\varepsilon-1}{\varepsilon+2} E_0 R^3$$

$$a_l = b_l = 0 \quad \text{for} \quad l \neq 1$$

Finally

$$\varphi_1 = -\frac{3E_0}{\varepsilon+2} r \cos\theta$$

$$\varphi_2 = -E_0 r \cos\theta + \frac{\varepsilon-1}{\varepsilon+2} \frac{E_0 R^3}{r^2} \cos\theta$$

A polarized sphere creates a potential that can be interpreted as the potential of a dipole (II-42) with a moment

$$p = 4\pi\varepsilon_0 \frac{\varepsilon-1}{\varepsilon+2} E_0 R^3.$$

44. Inside the sphere

$$\varphi_1 = \sum_{l=0}^{\infty} a_l r^l P_l(\cos\theta);$$

outside the sphere

$$\varphi_2 = \sum_{l=0}^{\infty} b_l r^{-l-1} P_l(\cos\theta)$$

We can see that $\varphi_1 = \varphi_2$ yields

$$b_l = a_l R^{2l+1}$$

From the condition that

$$\sigma = -\varepsilon_0 \left(\frac{\partial \varphi_2}{\partial r} - \frac{\partial \varphi_1}{\partial r} \right) \quad \text{for} \quad r = R$$

we find the relationship between the charge density σ and the unknown a_l's:

$$\sigma = \varepsilon_0 \sum_{l=0}^{\infty} a_l (2l+1) R^{l-1} P_l (\cos\theta)$$

We multiply this equation by $P_m (\cos\theta) \sin\theta$ and then integrate from 0 to π. Since $\sigma = 0$ for $0 < \theta < \alpha$,

$$a_l = \frac{\sigma}{2\varepsilon_0 R^{l-1}} \int_{\alpha}^{\pi} P_l (\cos\theta) \sin\theta \, d\theta$$

or

$$a_l = \frac{\sigma}{2\varepsilon_0 R^{l-1}} (2l+1)^{-1} [P_{l+1} (\cos\alpha) - P_{l-1} (\cos\alpha)]$$

where P_{l+1} is the Legendre polynomials, and $P_{l-1} (\cos\alpha) = -1$ for $l = 0$.

45. This problem can be solved by separating the variables in Cartesian coordinates:

$$\varphi (\mathbf{r}) = X (x) Y (y) Z (z) \qquad (1)$$

Substituting (1) in the Laplace equation, we get

$$\left. \begin{aligned} \frac{1}{X} \frac{d^2X}{dx^2} &= -\alpha^2 \\ \frac{1}{Y} \frac{d^2Y}{dy^2} &= -\beta^2 \\ \frac{1}{Z} \frac{d^2Z}{dz^2} &= \gamma^2 \end{aligned} \right\} \qquad (2)$$

where $\alpha^2 + \beta^2 = \gamma^2$.

Let a, b, and c be the lengths of the parallelepiped's edges. We assume that face $z = c$ is the one with the non-zero potential. Then the particular solution that satisfies the boundary conditions (i.e. all faces except $z = c$ have a zero potential) takes the form

$$\varphi_{nm} = \sin \alpha_n x \sin \beta_m y \sinh \gamma_{nm} z$$

where $\alpha_n = \frac{\pi n}{a}$, $\beta_m = \frac{\pi m}{b}$, $\gamma_{nm} = \pi \sqrt{\frac{n^2}{a^2} + \frac{m^2}{b^2}}$; n and m are integers.

The general solution can be found as a linear combination of particular solutions

$$\varphi(\mathbf{r}) = \sum_{n,\, m=1}^{\infty} A_{nm} \sin \alpha_n x \sin \beta_m y \sinh \gamma_{nm} z \tag{3}$$

The coefficients A_{nm} can be found from the condition that $\varphi(r) = V$ when $z = c$:

$$A_{nm} = \frac{4V}{ab \sinh \gamma_{nm} c} \int_0^a dx \int_0^b dy \sin \alpha_n x \sin \beta_m y$$

$$= \begin{cases} \dfrac{16V}{\pi^2 mn \sinh \gamma_{nm} c} & \text{for } n \text{ and } m \text{ odd} \\ 0 & \text{for } n \text{ or } m \text{ even} \end{cases}$$

Substituting A_{nm} into (3), we can find the potential at any point inside the parallelepiped.

46. $$\varphi(\mathbf{r}) = \sum_{n,\, m=1}^{\infty} \sin \alpha_n x \sin \beta_m y \left[A_{nm} \sinh \gamma_{nm} z + B_{nm} \cosh \gamma_{nm} z\right]$$

where

$$A_{nm} = \begin{cases} \dfrac{16V_2}{\pi^2 nm \sinh \gamma_{nm} c} - B_{nm} \coth \gamma_{nm} c & \text{for } n \text{ and } m \text{ odd} \\ 0 & \text{for } n \text{ or } m \text{ even} \end{cases}$$

$$B_{mn} = \begin{cases} \dfrac{16V_1}{\pi^2 nm} & \text{for } n \text{ and } m \text{ odd} \\ 0 & \text{for } n \text{ or } m \text{ even} \end{cases}$$

$$\alpha_n = \frac{\pi n}{a}, \quad \beta_m = \frac{\pi m}{b}, \quad \gamma_{nm} = \pi \sqrt{\frac{n^2}{a^2} + \frac{m^2}{b^2}}.$$

47. In cylindrical coordinates the solution of the Laplace equation can be represented in the following form:

$$\varphi(\rho, z) = R(\rho)\, Z(z)$$

where $R(\rho)$ and $Z(z)$ satisfy the equations

$$\frac{d^2 Z}{dz^2} = k^2 Z$$

$$\frac{d^2 R}{d\rho^2} + \frac{1}{\rho} \frac{dR}{d\rho} + k^2 R = 0$$

A particular solution of these equations is

$$\varphi(\rho, z) = \begin{cases} e^{k_l z} J_0(k_l \rho) & \text{for } z < 0 \\ e^{-k_l z} J_0(k_l \rho) & \text{for } z > 0 \end{cases}$$

where J_0 is the Bessel function of the first kind, and k_l is determined from the condition that on the surface of the cylinder the potential is zero, or

$$J_0(k_l r_0) = 0$$

The general solution can be found as a linear combination of particular solutions:

$$\varphi_1 = \sum_{l=1}^{\infty} A_l e^{k_l z} J_0(k_l \rho) \quad \text{for } z < 0$$

$$\varphi_2 = \sum_{l=1}^{\infty} A_l e^{-k_l z} J_0(k_l \rho) \quad \text{for } z > 0$$

The coefficients A_l can be found from the boundary condition on the surface of the disc. Using the condition (II-32) and the property of orthogonality of Bessel functions, we come to the relation

$$A_l = \frac{\sigma}{2\varepsilon_0 k_l} \frac{\int\limits_0^R \rho J_0(k_l \rho)\, d\rho}{\int\limits_0^{r_0} \rho J_0^2(k_l \rho)\, d\rho}$$

To find the potential of the point charge inside the cylinder we use the conditions for transforming to the limiting case: $R \to 0$, $\pi R^2 \sigma = e =$ constant. Then

$$A_l = \frac{e J_0(0)}{4\pi\varepsilon_0 k_l \int\limits_0^{r_0} J_0^2(k_l \rho)\, \rho\, d\rho}$$

Using the fact that $J_0(0) = 1$, we obtain the final expression for A_l:

$$A_l = \frac{e}{2\pi\varepsilon_0 k_l} \frac{1}{r_0^2 [J_1(k_l r_0)]^2}.$$

48. Let us choose the system of coordinates in such a way that the point charge is at the origin of coordinates and the z-axis is directed along the normal to the lamina. The potential for such a combination of charges and lamina consists of the potentials created by the charge and lamina separately. The potential of the polarized lamina satisfies the Laplace equation. Using its general solution in cylindrical coordinates, we can write

$$\varphi_1 = \frac{e}{4\pi\varepsilon_0 (\rho^2+z^2)^{1/2}} + \int_0^\infty B_1(k)\, e^{kz} J_0(k\rho)\, dk \quad (-\infty < z < d)$$

$$\begin{aligned} \varphi_2 &= \int_0^\infty A_2(k)\, e^{-kz} J_0(k\rho)\, dk \\ &\quad + \int_0^\infty B_2(k)\, e^{kz} J_0(k\rho)\, dk \quad (d < z < d+a) \end{aligned} \tag{1}$$

$$\varphi_3 = \int_0^\infty A_3(k)\, e^{-kz} J_0(k\rho)\, dk \quad (d+a < z < \infty)$$

The first term in φ_1 can be represented by an integral of a Bessel function

$$\frac{1}{(\rho^2+z^2)^{1/2}} = \int_0^\infty e^{-k|z|} J_0(k\rho)\, dk \tag{2}$$

Using the boundary conditions for the potential at $z = d$ and $z = d + a$, we find that

$$\begin{aligned} B_1(k) &= \frac{e\beta\,(e^{-2k(d+a)} - e^{-2kd})}{4\pi\varepsilon_0 (1-\beta^2 e^{-2ka})} \\ A_2(k) &= \frac{e\,(1-\beta)}{4\pi\varepsilon_0 (1-\beta^2 e^{-2ka})} \\ B_2(k) &= \frac{e\beta\,(1-\beta)\, e^{-2k(d+a)}}{4\pi\varepsilon_0 (1-\beta^2 e^{-2ka})} \\ A_3(k) &= \frac{e\,(1-\beta^2)}{4\pi\varepsilon_0 (1-\beta^2 e^{-2ka})} \end{aligned} \tag{3}$$

where $\beta = \dfrac{\varepsilon-1}{\varepsilon+1}$.

Formulas (1) and (3) give the solution of the problem.

If a charge is on the surface of a semi-infinite crystal, we must put $d \to 0$ and $a \to \infty$. In this case

$$B_1 = -\frac{e\beta}{4\pi\varepsilon_0}, \quad B_2 = 0, \quad A_2 = \frac{e}{2\pi\varepsilon_0(\varepsilon+1)}$$

Substituting these expressions in (1) and using the expansion (2), we obtain

$$\varphi_1 = \varphi_2 = \frac{e}{2\pi\varepsilon_0(\varepsilon+1)\,r} \tag{4}$$

where $r = (\rho^2 + z^2)^{1/2}$ is the distance from the charge to the observation point. Formula (4) coincides with the respective expression in Problem 36.

49. The symmetry of the problem implies that the axes of the ellipsoid are the principal axes of the quadrupole moment tensor. In the set of principal axes

$$D_{xy} = D_{yz} = D_{zx} = 0$$

Changing the variables, $x' = x'a$, $y = y'b$, $z = z'c$, we reduce the integration over the volume of the ellipsoid in Eq. (II-44) to an integration over the volume of the sphere $x'^2 + y'^2 + z'^2 = 1$.

Equation (II-44) yields

$$D_{xx} = \frac{e}{5}(2a^2 - b^2 - c^2)$$

$$D_{yy} = \frac{e}{5}(2b^2 - a^2 - c^2)$$

$$D_{zz} = \frac{e}{5}(2c^2 - a^2 - b^2)$$

where $e = \frac{4\pi}{3} abc\rho$ is the charge of the ellipsoid.

50. Let us solve the equations

$$\begin{aligned} \operatorname{div} \mathbf{D} &= e\delta(\mathbf{r}) \\ \operatorname{curl} \mathbf{E} &= 0 \end{aligned} \tag{1}$$

assuming that charge e is at the origin of coordinates.

We direct the Cartesian axes along the principal axes of the permittivity tensor. Then

$$D_x = \varepsilon_x \varepsilon_0 E_x = -\varepsilon_x \varepsilon_0 \frac{\partial \varphi}{\partial x}$$
$$D_y = \varepsilon_y \varepsilon_0 E_y = -\varepsilon_y \varepsilon_0 \frac{\partial \varphi}{\partial y} \tag{2}$$
$$D_z = \varepsilon_z \varepsilon_0 E_z = -\varepsilon_z \varepsilon_0 \frac{\partial \varphi}{\partial z}$$

Substitute (2) into (1) and get

$$\varepsilon_x \frac{\partial^2 \varphi}{\partial x^2} + \varepsilon_y \frac{\partial^2 \varphi}{\partial y^2} + \varepsilon_z \frac{\partial^2 \varphi}{\partial z^2} = -\frac{e}{\varepsilon_0} \delta(\mathbf{r}) \tag{3}$$

Changing the variables, $x' = \frac{x}{\sqrt{\varepsilon_x}}$, $y' = \frac{y}{\sqrt{\varepsilon_y}}$, $z' = \frac{z}{\sqrt{\varepsilon_z}}$, we reduce Eq. (3) to the following:

$$\frac{\partial^2 \varphi}{\partial x'^2} + \frac{\partial^2 \varphi}{\partial y'^2} + \frac{\partial^2 \varphi}{\partial z'^2} = -\frac{e}{\varepsilon_0 (\varepsilon_x \varepsilon_y \varepsilon_z)^{1/2}} \delta(\mathbf{r}') \tag{4}$$

where we have used a property of δ-function:

$$\delta(ax) = \frac{1}{a} \delta(x)$$

Equation (4) has a solution

$$\varphi = \frac{e}{4\pi\varepsilon_0 (\varepsilon_x \varepsilon_y \varepsilon_z)^{1/2}} \frac{1}{r'}$$

where $r' = \left(\frac{x^2}{\varepsilon_x} + \frac{y^2}{\varepsilon_y} + \frac{z^2}{\varepsilon_z} \right)^{1/2}$.

51. Direct the y-axis along the normal to the lamina. From the boundary conditions $E_{1t} = E_{2t}$, $D_{1n} = D_{2n}$ it follows that

$$\mathbf{E} = \mathbf{E}_0 + \frac{\varepsilon_0 E_y - \varepsilon_{yx} E_x - \varepsilon_{yy} E_y - \varepsilon_{yz} E_z}{\varepsilon_{yy}} \mathbf{n}$$

where $\mathbf{n}$ is the unit normal vector.

52. $W = -\frac{e^2}{4\pi\varepsilon_0 a}$.

53. $W = \frac{e_1 e_2}{4\pi\varepsilon_0 a}$.

54. If we use the answer to Problem 36, we find that

$$e = R(1+\varepsilon)\sqrt{\frac{4\pi\varepsilon_0}{\varepsilon-1}\left(\frac{2}{3}\pi R^3\mu g - Mg\right)}$$

where g is the acceleration due to gravity.

55. $I = \frac{\varphi_1 - \varphi_2}{R}$, $R = \frac{1}{4\pi\sigma^*}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$.

56. $\frac{\tan\alpha_1}{\tan\alpha_2} = \frac{\sigma_1^*}{\sigma_2^*}$, where α_1 and α_2 are the angles between the lines of current and the normal to the boundary in medium *1* and medium *2*, respectively.

57. The potential difference between the electrodes satisfies the Poisson equation

$$\Delta\varphi = -\frac{\rho}{\varepsilon_0} \tag{1}$$

where $\rho = j/v$, j is the current density, which in the stationary case does not depend on x if the x-axis is directed along the motion of charges. Velocity v can be found from the law of conservation of energy

$$\frac{mv^2}{2} + e\varphi = 0$$

We can rewrite Eq. (1) in the following form:

$$\frac{d^2\varphi}{dx^2} = -\frac{j\sqrt{m}}{\varepsilon_0\sqrt{2|e|\varphi}}$$

Solving this equation under the conditions that $\varphi = 0$ for $x = 0$, $\varphi = -V$ for $x = d$, and $\frac{\partial\varphi}{\partial x} = 0$ for $x = 0$ (the last condition implies that the electric field near the first electrode is zero), we get

$$j = -\frac{4\varepsilon_0}{9d^2}\sqrt{\frac{2|e|}{m}}|V|^{3/2}.$$

58. $\mathbf{H} = \frac{1}{2}[\mathbf{j}\times\mathbf{r}]$ for $r < R$

$\mathbf{H} = \frac{R^2}{2r^2}[\mathbf{j}\times\mathbf{r}]$ for $r > R$

where r is the distance from the axis of the cylinder to the observation point.

59. $\mathbf{H} = \frac{1}{2} [\mathbf{j} \times \mathbf{a}]$.

60. Let the z-axis be directed along the axis of the conductor. The symmetry of the problem yields

$$A_x = A_y = 0$$
$$\Delta A_{1z} = -\mu_0 j \quad \text{for} \quad \rho < R$$
$$\Delta A_{2z} = 0 \quad \text{for} \quad \rho > R$$

The A_z-component depends only on the distance from the axis. In cylindrical coordinates

$$\frac{1}{\rho} \frac{d}{d\rho} \left(\rho \frac{dA_{1z}}{d\rho} \right) = -\mu_0 \frac{a}{\rho} \quad \text{for} \quad \rho < R$$
$$\frac{1}{\rho} \frac{d}{d\rho} \left(\rho \frac{dA_{2z}}{d\rho} \right) = 0 \quad \text{for} \quad \rho > R$$

Solving these equations, we get

$$A_{1z} = -\mu_0 a\rho + C_1 \ln \rho + C_2 \tag{1}$$
$$A_{2z} = C_3 \ln \rho + C_4 \tag{2}$$

Considering that $\frac{dA_{1z}}{d\rho} \to \infty$ as $\rho \to 0$ and that in this case the magnetic field tends to infinity, we must put C_1 equal to zero. The continuity of A_z and $\frac{dA_z}{d\rho}$ implies (when there are no surface currents) that

$$C_3 = -\mu_0 aR, \quad C_4 = -\mu_0 aR\,(1 - \ln R) + C_2 \tag{3}$$

The relations (1)-(3) define the vector potential up to a constant term.

The magnetic field strength is defined by the following formulas:

$$H_{1x} = -a \sin\varphi, \quad H_{1y} = a\cos\varphi, \quad H_{1z} = 0 \quad \text{or} \quad \mathbf{H}_1 = \frac{1}{\rho} [\mathbf{a} \times \boldsymbol{\rho}]$$

where vector $\mathbf{a}$ is directed along the axis of the conductor and in magnitude is equal to a. The absolute value of $\mathbf{H}_1$ is a and is the same everywhere inside the conductor.

Outside the conductor

$$H_{2x} = -\frac{aR}{\rho} \sin\varphi, \quad H_{2y} = \frac{aR}{\rho} \cos\varphi, \quad H_{2z} = 0 \quad \text{or} \quad |\mathbf{H}_2| = \frac{aR}{\rho}.$$

61. Let us choose the system of coordinates so that the z-axis is directed along the current and the x-axis along the normal to the plane. From the boundary condition (II-26) we then have

$$H_y = \begin{cases} -\frac{i}{2} & \text{for} \quad x < 0, \\ \frac{i}{2} & \text{for} \quad x > 0. \end{cases}$$

62. (a) $H = 0$ between the planes,
$H = i$ outside the planes;
(b) $H = i$ between the planes,
$H = 0$ outside the planes.

63. Let us place the origin of coordinates on the midline of the strip, and direct the z-axis along the strip and the x-axis normal to its surface. The magnetic field does not, obviously, depend on the z-coordinate. We mentally divide the strip into strips parallel to the z-axis. The width of these strips must be so small that in each one the current can be considered linear. Let the position of any such strip be y', and its width dy'. According to Problem 58, at a point with coordinates x, y this strip will generate a magnetic field of intensity $d\mathbf{H}$ that is determined by the following expressions:

$$dH_x = -\frac{i(y-y')\,dy'}{2\pi[x^2+(y-y')^2]}$$

$$dH_y = \frac{ix\,dy'}{2\pi[x^2+(y-y')^2]}$$

$$dH_z = 0$$

Integrating these relations from $-a/2$ to $a/2$, we get

$$H_x = \frac{i}{4\pi}\ln\frac{x^2+(y-a/2)^2}{x^2+(y+a/2)^2}$$

$$H_y = \frac{i}{2\pi}\left(-\arctan\frac{y-a/2}{x}+\arctan\frac{y+a/2}{x}\right)$$

$$H_z = 0$$

As $a \to \infty$ (the limiting case of a conducting plane),

$$H_x = 0, \quad H_y = \begin{cases} -\dfrac{i}{2} & \text{for} \quad x < 0 \\ \dfrac{i}{2} & \text{for} \quad x > 0 \end{cases}$$

which coincides with the answer to Problem 61.

64. For linear conductors of finite length l parallel to the z-axis

$$A_z = \frac{\mu_0 I}{4\pi}\left(\int_{-l}^{l} \frac{dz}{\rho_1} - \int_{-l}^{l} \frac{dz}{\rho_2}\right)$$

$$A_x = A_y = 0$$

where $\rho_1 = (z^2 + r_1^2)^{1/2}$ and $\rho_2 = (z^2 + r_2^2)^{1/2}$; r_1 and r_2

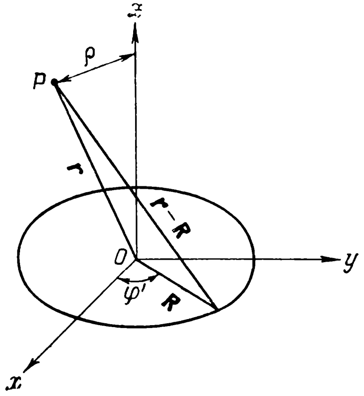

Fig. 56

are the distances from the observation point to the first and second conductors, respectively.

After evaluating the integrals and passing to the limit as $l \to \infty$, we get

$$A_z = \frac{\mu_0 I}{2\pi} \ln \frac{r_2}{r_1}.$$

65. We direct the z-axis along the normal to the plane of the ring (Fig. 56). Since the problem is cylindrically symmetric, we can assume that the xz-plane passes through

the observation point P. The vector potential is directed along the y-axis, i.e. has only one nonzero component

$$A_\varphi = \frac{\mu_0 I}{4\pi} \oint \frac{dl_{\varphi'}}{|\mathbf{r}-\mathbf{r}'|} = \frac{\mu_0 I}{4\pi} \int_0^{2\pi} \frac{R\cos\varphi'\, d\varphi'}{(R^2+\rho^2+z^2-2R\rho\cos\varphi')^{1/2}}$$

where ρ is the distance from the observation point P to the axis of the ring. If we put $\varphi' = \pi + 2\theta$, we find that

$$A_\varphi = \frac{\mu_0 RI}{\pi} \int_0^{\pi/2} \frac{(2\sin^2\theta - 1)\, d\theta}{[(R+\rho)^2+z^2-4R\rho\sin^2\theta]^{1/2}}$$

If we introduce the variable $k^2 = \frac{4R\rho}{(R+\rho)^2+z^2}$, by using straightforward calculations we come to the expression for A_φ:

$$A_\varphi = \frac{\mu_0 I}{k\pi} \left(\frac{R}{\rho}\right)^{1/2} \left[\left(1-\frac{1}{2}k^2\right) K(k) - E(k)\right]$$

where $K(k) = \int_0^{\pi/2} \frac{d\theta}{(1-k^2\sin^2\theta)^{1/2}}$ is an elliptic integral of the first kind, and $E(k) = \int_0^{\pi/2} (1-k^2\sin^2\theta)^{1/2}\, d\theta$ an elliptic integral of the second kind. The two integrals are related in the following manner:

$$\frac{dK}{dk} = \frac{E}{k(1-k^2)} - \frac{K}{k}, \quad \frac{dE}{dk} = \frac{E}{k} - \frac{K}{k}$$

Using these relationships, we get the final expression for the magnetic field strength:

$$H_\rho = \frac{I}{2\pi} \frac{z}{\rho[(R+\rho)^2+z^2]^{1/2}} \left[-K(k) + \frac{R^2+\rho^2+z^2}{(R-\rho)^2+z^2} E(k)\right]$$

$$H_z = \frac{I}{2\pi} \frac{1}{[(R+\rho)^2+z^2]^{1/2}} \left[K(k) + \frac{R^2-\rho^2-z^2}{(R-\rho)^2+z^2} E(k)\right]$$

$$H_\varphi = 0$$

On the axis of the ring $\rho \to 0$, and so

$$H_\rho \to 0, \quad H_z = \frac{R^2 I}{2(R^2+z^2)^{3/2}}, \quad H_\varphi = 0.$$

66. Outside the ball $B_2 = \mu_0 H_2$, $\operatorname{curl} \mathbf{H}_2 = 0$, and $\operatorname{div} \mathbf{H}_2 = 0$. This means that the magnetic field strength can be represented by the gradient of a scalar function φ_m, which satisfies the Laplace equation $\Delta\varphi_m = 0$, i.e.

$$\mathbf{H}_2 = -\nabla\varphi_m \tag{1}$$

The general solution for a potential that generates a zero magnetic field at infinity is then

$$\varphi_m = \sum_{l=0}^{\infty} a_l \frac{P_l(\cos\theta)}{r^{l+1}} \tag{2}$$

Inside the ball $\mathbf{H}_1$, $\mathbf{B}_1$, and $\mathbf{M}$ are parallel. Let the z-axis be in the direction of these vectors. Using the boundary conditions (the continuity of B_r and H_θ at $r = R$), we obtain

$$B_1 \cos\theta = \mu_0 \sum_{l=0}^{\infty} \frac{(l+1)\, a_l P_l(\cos\theta)}{R^{l+2}}$$

$$H_1 \sin\theta = \sum_{l=0}^{\infty} \frac{a_l}{R^{l+2}} \frac{dP_l(\cos\theta)}{d\theta}$$

Whence, the nonzero coefficients a_l are those with $l = 1$. Substituting $\frac{\mathbf{B}_1}{\mu_0} - \mathbf{M}$ for $\mathbf{H}_1$, we find the equations for a_1 and $\mathbf{M}$

$$B_1 = \mu_0 \frac{2a_1}{R^3}, \quad M - \frac{B_1}{\mu_0} = \frac{a_1}{R^3}$$

Solving them we get

$$a_1 = \frac{1}{3} M R^3, \quad B_1 = \frac{2\mu_0}{3} M \tag{3}$$

From relations (1)-(3) we get the expressions for the sought quantities outside the ball

$$\mathbf{H}_2 = \frac{R^3}{3}\left[\frac{3\mathbf{r}(\mathbf{M}\cdot\mathbf{r})}{r^5} - \frac{\mathbf{M}}{r^3}\right], \quad \mathbf{B}_2 = \mu_0 \mathbf{H}_2$$

Thus, outside the ball the magnetic field is the field of a dipole that has a magnetic moment

$$m=\frac{4\pi}{3}R^3M$$

Inside the ball

$$\mathbf{B}_1=\frac{2\mu_0}{3}\mathbf{M},\quad \mathbf{H}_1=-\frac{1}{3}\mathbf{M}.$$

67. Since no electrical currents are present, we can introduce a scalar potential φ_m which satisfies the relationship $\mathbf{H} = -\nabla\varphi_m$ and which can be determined from the Laplace equation $\Delta\varphi_m = 0$. The general solution of this equation is

$$\varphi_m(r)=-\frac{1}{4\pi}\int\frac{\operatorname{div}\mathbf{M}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}dV'-\frac{1}{4\pi}\oint\frac{M_n}{|\mathbf{r}-\mathbf{r}'|}dS' \qquad (1)$$

where $\mathbf{M}(\mathbf{r})$ is the magnetization vector of the magnet, and S' is any closed surface that contains the magnet.

In our case $\mathbf{M}$ = constant and $\operatorname{div}\mathbf{M} = 0$, and M_n is nonzero only on the bases of the magnet. If we choose the z-axis directed along the cylinder's axis, we get

$$\varphi_m=\frac{M_0}{4\pi}\int\frac{dx\,dy}{[(z+d/2)^2+x^2+y^2]^{1/2}}-\frac{M_0}{4\pi}\int\frac{dx\,dy}{[(z-d/2)^2+x^2+y^2]^{1/2}} \qquad (2)$$

After integrating (2), we get

$$\varphi_m=\frac{M_0}{2}\left[\sqrt{(z+d/2)^2+R^2}-|z+d/2|-\sqrt{(z-d/2)^2+R^2}+|z-d/2|\right] \qquad (3)$$

Inside the magnet

$$H_z=-\frac{\partial\varphi_m}{\partial z}=\frac{M_0}{2}\left(\frac{z+d/2}{\sqrt{(z+d/2)^2+R^2}}+\frac{z-d/2}{\sqrt{(z-d/2)^2+R^2}}-2\right).$$

68. $M_x=M_y=0,\ M_z=\frac{e\Omega}{5}R^2.$

69. The rotary motion of the sphere generates a surface current in it. In spherical coordinates with the polar axis directed along the axis of rotation, this current is

$$i_\varphi=\sigma\Omega R\sin\theta \qquad (1)$$

Since there are no currents either inside or outside the sphere, we can introduce a magnetic potential, i.e.

$$\mathbf{H}=\begin{cases} -\nabla\psi_1 & \text{when} \quad r<R \\ -\nabla\psi_2 & \text{when} \quad r>R \end{cases}$$

where

$$\psi_1=\sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) \tag{2}$$

$$\psi_2=\sum_{l=0}^{\infty} B_l r^{-(l+1)} P_l(\cos\theta) \tag{3}$$

The boundary conditions (II-25) and (II-26) for our case imply that, for $r = R$,

$$\frac{\partial\psi_1}{\partial r}=\frac{\partial\psi_2}{\partial r}, \quad \frac{\partial\psi_1}{r\,\partial\theta}-\frac{\partial\psi_2}{r\,\partial\theta}=i_\varphi$$

Substituting expansions (2) and (3) and formula (1) into the boundary conditions, and having in mind that

$$\sin\theta=-\frac{d(\cos\theta)}{d\theta}=-\frac{dP_1(\cos\theta)}{d\theta}$$

we obtain

$$A_1=-\frac{2}{3}\sigma\Omega R, \quad B_1=\frac{1}{3}\sigma\Omega R^4$$

$$A_l=B_l=0 \quad \text{for} \quad l\neq 1$$

Thus,

$$\mathbf{H}=\frac{2}{3}\sigma\Omega R \quad \text{for} \quad r<R$$

(i.e. inside the sphere the field is directed along the z-axis);

$$\mathbf{H}=\frac{3(\mathbf{m}\cdot\mathbf{r})\,\mathbf{r}}{4\pi r^5}-\frac{\mathbf{m}}{4\pi r^3} \quad \text{for} \quad r>R$$

$$m=\frac{4\pi}{3}R^4\sigma\Omega$$

i.e. outside the sphere the magnetic field is the field of a system with a magnetic moment $\mathbf{m}$.

70. The force per unit length is

$$F=\pm\frac{\mu\mu_0}{2\pi}\frac{I_1 I_2}{d}$$

Here the "plus" sign is used when both currents flow in the same direction, and the "minus" when they flow in opposite directions.

71. The symmetry of the problem implies that the force is an attractive force. The radial component of the magnetic field, which is constant everywhere in the contour, is the sole contributor to this force. And so

$$F=\mu\mu_0 I_1 H_\rho (R_1, R_2, d)\int_0^{2\pi} R_1\,d\theta = 2\pi\mu\mu_0 R_1 I_1 H_\rho (R_1, R_2, d)$$

where H_ρ is the radial component of the magnetic field strength that the current in contour *2* creates in contour *1*.

Using the conditions of Problem 65, we come to the final expression for F:

$$F=\frac{\mu\mu_0 I_1 I_2 d}{[(R_1+R_2)^2+d^2]^{1/2}}\left[-K(k)+\frac{R_1^2+R_2^2+d^2}{(R_1-R_2)^2+d^2}E(k)\right]$$

where $k^2=\dfrac{4R_1R_2}{(R_1+R_2)^2+d^2}$.

72. The magnetic field between the coaxial conductors is

$$H=\frac{I}{2\pi r}$$

where I is the current in the conductor.

On the one hand the energy of the magnetic field is

$$W=\frac{1}{2}\int BH\,dV=\frac{\mu\mu_0 I^2}{4\pi}\ln\frac{R_2}{R_1}$$

On the other hand,

$$W=\frac{1}{2}LI^2$$

Comparing the two equations, we find that

$$L=\frac{\mu\mu_0}{2\pi}\ln\frac{R_2}{R_1}.$$

73. $L=\dfrac{\mu_1\mu_0}{8\pi}+\dfrac{\mu_2\mu_0}{2\pi}\ln\dfrac{R_2}{R_1}.$

74. $\chi=-\dfrac{\mu_0 e^2 a^2}{2m}N.$

76. $E = aJ_0(kr)\,e^{-i\omega t}$ and $H = -ai\sqrt{\dfrac{\sigma^* i}{\omega\mu_0}}\,J_1(kr)\,e^{-i\omega t}$, where $k = \dfrac{1+i}{\sigma}$, $\sigma = \left(\dfrac{2}{\omega\sigma^*\mu_0}\right)^{1/2}$, and J_0 and J_1 are Bessel functions.

77. The time dependence of the magnetization vector is given by the following equation:

$$\frac{d\mathbf{M}}{dt} = \mu_0 g\,[\mathbf{M}\times\mathbf{H}] \tag{1}$$

where g is the gyromagnetic ratio.

The magnetic field strength for the given system can be written in the form

$$\mathbf{H} = \mathbf{H}_0 + \mathbf{h}e^{-i\omega t}$$

Let us look for the solution of (1) in the form

$$\mathbf{M} = \mathbf{M}_0 + \mathbf{m}e^{-i\omega t}$$

where $\mathbf{m}$ is the additional magnetization induced by the variable field.

Since $h \ll H_0$, we conclude that $m \ll M_0$. If we ignore infinitesimals of order hm, Eq. (1) yields

$$\begin{aligned} -i\omega m_x &= -\mu_0 g M_0 h_y + \mu_0 g m_y H_0 \\ -i\omega m_y &= \mu_0 g M_0 h_x - \mu_0 g m_x H_0 \\ -i\omega m_z &= 0 \end{aligned}$$

After solving these equations we get

$$\begin{aligned} m_x &= \chi h_x - i\nu h_y \\ m_y &= i\nu h_x + \chi h_y \\ m_z &= 0 \end{aligned} \tag{2}$$

where $\chi = gM_0\mu_0\dfrac{\omega_0}{\omega^2-\omega_0^2}$, $\nu = gM_0\mu_0\dfrac{\omega}{\omega^2-\omega_0^2}$, $\omega_0 = -\mu_0 g H_0$. When $\omega \cong \omega_0$, the effect of magnetic resonance is observed.

78. If the wavelength is considerably greater than the size of the sample, we can use the magnetostatic approximation to describe the oscillations of the magnetic moment. In this approximation the varying field $\mathbf{h}$ and the magnetic moment $\mathbf{m}$ that are generated by the oscillations satisfy

the following equations:

$$\operatorname{curl} \mathbf{h} = 0$$

$$\operatorname{div} (\mathbf{h} + \mathbf{m}) = 0$$

For a sample in the form of a ball these equations are solved in Problem 66. The solution implies that inside the ball

$$\mathbf{h} = -\frac{1}{3}\mathbf{m}$$

On the other hand, **h** and m are also linked by the relationship (2) of Problem 77. Because of this

$$h_x = -\frac{1}{3}(\chi h_x - i\nu h_y)$$

$$h_y = -\frac{1}{3}(\chi h_y + i\nu h_x)$$

The system of equations has a solution if its determinant is zero, which places a restriction on the value of the frequency of oscillation ω. Non-trivial solutions exist if

$$\omega = \pm g\left(H_0 + \frac{1}{3}M_0\right)$$

where $\mathbf{M}_0$ is static magnetization created in the ball by the field $\mathbf{H}_0$.

79. In the case of a nonconducting medium and for a slowly varying magnetic field **h** the system of Eqs. (II-58) is of the form

$$\operatorname{curl} \mathbf{h} = 0 \tag{1}$$

$$\operatorname{div} (\mathbf{h} + \mathbf{m}) = 0 \tag{2}$$

where **m** is the varying magnetic moment.

Equation (1) implies that a magnetic potential φ_m can be introduced, so that $\mathbf{h} = -\nabla\varphi_m$. Substituting the last equality into (2) and keeping in mind that, according to Problem 77, m is a function of **h**, we come to the following equation for φ_m:

$$\mu\left(\frac{\partial^2\varphi_m}{\partial x^2} + \frac{\partial^2\varphi_m}{\partial y^2}\right) + \frac{\partial^2\varphi_m}{\partial z^2} = 0 \tag{3}$$

For an infinite medium we seek the solution of (3) in the form $\varphi_m = \varphi_0 e^{i\mathbf{kr}}$. We then have $\mu(k_x^2 + k_y^2) + k_z^2 = 0$

or $\tan^2\theta_k = \mu$, where θ_k is the angle between $\mathbf{k}$ and the z-axis, i.e. between $\mathbf{k}$ and the direction of the constant magnetic field H_0. Using the value of χ from Problem 77, we get the frequency of natural oscillations as a function of the direction of wave vector $\mathbf{k}$:

$$\omega^2 = \omega_0(\omega_0 + \omega_M \cos^2\theta_k), \quad \text{where} \quad \omega_M = -g\mu_0\mu$$

Thus the spectrum of possible frequencies is

$$\omega_0 < \omega < \sqrt{\omega_0(\omega_0 + \omega_M)}$$

We must note that the obtained result is only valid in the domain $k \gg \frac{\omega}{c}$, i.e. when it is possible to use the magnetostatic approximation and when in Maxwell's equations the term $\frac{\partial \mathbf{E}}{\partial t}$ can be neglected.

80. The magnetic potential φ_m for the given system satisfies Eq. (3) of Problem 79. For an ideally conducting covering, the boundary condition is of the form $B_n = 0$, or

$$\frac{\partial \varphi_m}{\partial z} = 0 \quad \text{for} \quad z = 0 \quad \text{and} \quad z = d \tag{1}$$

with the z-axis directed along the normal to the surface of the covering.

Let us examine the waves that propagate along the y-axis. The solution of Eq. (3) of Problem 79 can be written in the form

$$\varphi_m = (A \cos k_z z + B \sin k_z z)\, e^{ik_y y}$$

where

$$k_z^2 = -\mu k_y^2 \tag{2}$$

From the boundary condition (1) it follows that $A = 0$ and $k_z = \pi n/d$, where n is an integer. By substituting these results into (2) and using the dependence of μ on ω, we get the dispersion equation for magnetostatic waves in the plate

$$\omega^2 = \omega_0\left(\omega_0 + \frac{\omega_M}{1+\left(\frac{\pi n}{k_y d}\right)^2}\right).$$

81. By analogy with Problem 80 we must also examine the solution outside the plate. At the boundary this solution

must be fitted to the solution inside the plate:

$$\varphi_m^{\text{inside}} = \varphi_m^{\text{outside}}, \quad \left.\frac{\partial \varphi_m}{\partial z}\right|_{\text{inside}} = \left.\frac{\partial \varphi_m}{\partial z}\right|_{\text{outside}}$$

for $z = 0$ and $z = d$.

For the wave travelling along the y-axis the frequency spectrum can be determined from the conditions that

$$\tan k_z d = \frac{2k_y k_z}{k_y^2 - k_z^2}, \quad k_z^2 = -\mu k_y^2 \tag{1}$$

The system of equations (1) gives the dependence of ω on k_y.

82. $\omega_{12}^2 = \dfrac{L_1C_1 + L_2C_2 \mp [(L_1C_1 - L_2C_2)^2 + 4C_1C_2L_{12}^2]^{1/2}}{2C_1C_2(L_1L_2 - L_{12}^2)}$.

83. The head of the resultant electric-field vector describes an ellipse with the semiaxes

$$a = \sqrt{E_{01}^2 \cos^2\alpha + E_{02}^2 \cos^2(\alpha - \varphi)}$$

$$b = \sqrt{E_{01}^2 \sin^2\alpha + E_{02}^2 \sin^2(\alpha - \varphi)}$$

where $\tan 2\alpha = \dfrac{E_{02}^2 \sin 2\varphi}{E_{01}^2 + E_{02}^2 \cos 2\varphi}$. The principal axes of the ellipse are rotated through an angle α in relation to vectors $\mathbf{E}_1$ and $\mathbf{E}_2$.

84. The relationship between the angle of incidence, θ_1, and the angle of refraction, θ_2, is

$$\sin\theta_2 = \frac{\sin\theta_1}{n_{12}} \tag{1}$$

where $n_{12} = \sqrt{\varepsilon_2/\varepsilon_1}$ is the index of refraction of the second medium relative to the first. If ε_2 is less than ε_1, relation (1) holds true if we choose θ_2 complex. The electromagnetic wave in the medium from which total internal reflection occurs (when $\varepsilon_2 < \varepsilon_1$) is given by the formula

$$\mathbf{E}_2 = \mathbf{E}_{02} e^{i\mathbf{kr} - i\omega t}$$

where $k = \sqrt{\varepsilon_2}/c$, and ω is the frequency of the wave.

If we choose the coordinate system in such a way that the interface of the two media lies in the xy-plane and the wave

vector **k** in the xz-plane, then

$$\mathbf{E}_2 = \mathbf{E}_{02} e^{-i\omega\left(t - \frac{x \sin\theta_2 + z\cos\theta_2}{v_2}\right)} \tag{2}$$

where $v_2 = c/\sqrt{\varepsilon_2}$ is the phase velocity of the wave. Let us express angle θ_2 in terms of angle θ_1 using the relationship (1):

$$\cos\theta_2 = \pm\sqrt{1-\sin^2\theta_2} = \pm i\sqrt{\frac{\sin^2\theta_1}{n_{12}} - 1} \tag{3}$$

where we must choose the "minus" sign for the field to be finite as $z \to \infty$. Substituting (3) into (2), we get

$$\mathbf{E}_2 = \mathbf{E}_{02} e^{-i\omega\left(t - \frac{x\sin\theta_1}{v_1}\right) - \varkappa z}$$

where $\varkappa = \frac{\omega}{v_2}\left(\frac{\sin^2\theta_1}{n_{12}^2} - 1\right)^{1/2}$.

Thus, the wave in the medium from which total internal reflection occurs is a wave that travels along the interface and that is also damped in the reflecting media with a logarithmic decrement $\varkappa$. The depth of penetration is

$$d = \frac{1}{\varkappa} = \frac{v_2}{\omega}\left(\frac{\sin^2\theta_1}{n_{12}^2} - 1\right)^{-1/2}.$$

85. $R = \frac{(1-n)^2+\varkappa^2}{(1+n)^2+\varkappa^2}$. Here

$$n = \sqrt{\frac{\varepsilon}{2}}\left[\sqrt{1+\left(\frac{\sigma^*}{\omega\varepsilon\varepsilon_0}\right)^2} + 1\right]^{1/2}$$

$$\text{and}\quad \varkappa = \sqrt{\frac{\varepsilon}{2}}\left[\sqrt{1+\left(\frac{\sigma^*}{\omega\varepsilon\varepsilon_0}\right)^2} - 1\right]^{1/2}$$

In the case of an ideal conductor ($\varepsilon \ll \sigma^*/\omega$),

$$R = 1 - 2\left(\frac{2\varepsilon_0\omega}{\sigma^*}\right)^{1/2}.$$

86. The electric field can be found from the equation

$$\frac{d^2E}{dz^2} + k^2E = 0 \tag{1}$$

where $k = \frac{\sqrt{\varepsilon}}{c}$ inside the plate and $k = k_0 = \frac{\omega}{c}$ outside the plate. The z-axis is directed along the normal to the surface of the plate.

The solution of Eq. (1) in the region with the incident and reflected waves has the form

$$E_1 = E_0 e^{ik_0 z} + A e^{-ik_0 z}$$

where E_0 is the amplitude of the incident wave, and A that of the reflected wave.

Inside the plate the solution is

$$E_2 = E_+ e^{ikz} + E_- e^{-ikz}$$

In the region with only the transmitted wave

$$E_3 = D e^{ik_0 z}$$

where D is the amplitude of the transmitted wave.

The boundary conditions, i.e. the continuity of the tangential component of the electric field vector, bring us to the following equations for the electric fields in all three regions:

$$E_0 + A = E_+ + E_-$$

$$E_+ e^{ikd} + E_- e^{-ikd} = D e^{ik_0 d}$$

$$E_0 - A = nE_+ - nE_-$$

$$nE_+ e^{ikd} - nE_- e^{-ikd} = D e^{ik_0 d}$$

where $n = \sqrt{\varepsilon}$ is the plate's index of refraction.

Solving this system of equations, we get

$$A = \frac{\sqrt{\rho_0}\,(1 - e^{2ikd})}{1 - \rho_0 e^{2ikd}} E_0$$

$$E_+ = \frac{2}{(1+n)(1 - \rho_0 e^{2ikd})} E_0$$

$$E_- = -\frac{2\rho_0}{(1-n)(e^{-2ikd} - \rho_0)} E_0$$

$$D = \frac{\delta_0 e^{ikd}}{1 - \rho_0 e^{2ikd}} E_0$$

where $\rho_0 = \left(\frac{1-n}{1+n}\right)^2$ and $\delta_0 = \frac{4n}{(1+n)^2}$ are, respectively, the reflectance and transmittance for a semi-infinite medium.

Using the obtained relationships, we find the reflectance of electromagnetic waves from a plane-parallel plate:

$$\rho = \frac{|A|^2}{|E_0|^2} = \frac{4\rho_0 \sin^2 kd}{\delta_0^2 + 4\rho_0 \sin^2 kd}$$

There is no reflection if

$$d = \frac{\pi m}{k} = \frac{m\lambda}{2}$$

where m is an integer, and λ the wavelength inside the plate.

87. Choose the x-axis directed along the line of propagation of the wave, the y-axis along the wave's magnetic field strength, and the xy-plane in the interface of the two dielectrics. Suppose the medium with permittivity ε_1 lies in the region $z > 0$, and the one with permittivity $-|\varepsilon_2|$ in the region $z < 0$. The solution of the equation

$$\Delta H_y + \frac{\omega^2}{c^2} \varepsilon H_y = 0$$

which describes a wave that travels along the interface and also dissipates far away from the interface has the following form:

$$\begin{aligned} H_{1y} &= H_{01} e^{ikx - \varkappa_1 z}, \text{ where } \varkappa_1 = \sqrt{k^2 - \omega^2 \varepsilon_1 / c^2} \quad \text{for } z > 0 \\ H_{2y} &= H_{02} e^{ikx + \varkappa_2 z}, \text{ where } \varkappa_2 = \sqrt{k^2 + \omega^2 |\varepsilon_2| / c^2} \quad \text{for } z < 0 \end{aligned} \qquad (1)$$

The boundary conditions $H_{1y} = H_{2y}$, $E_{1x} = E_{2x}$ (for $z = 0$) yield

$$H_{01} = H_{02}, \quad \varkappa_1/\varepsilon_1 = \varkappa_2/|\varepsilon_2| \qquad (2)$$

The last expression holds only if $\varepsilon_1 < |\varepsilon_2|$. Eliminating $\varkappa_1$ and $\varkappa$, with the help of formula (1), we obtain

$$k^2 = \frac{\omega^2 \varepsilon_1 |\varepsilon_2|}{c^2 (|\varepsilon_2| - \varepsilon_1)}.$$

88. The equality of the projections of the wave vectors on the interface yields (a) for the ordinary ray

$$\frac{\sin \theta_2'}{\sin \theta_1} = (\varepsilon_\perp)^{1/2}$$

and (b) for the extraordinary ray

$$k_1 \sin \theta_1 = k_2 \sin \theta_2'' = k_0 \sin \theta_2'' \left(\frac{\varepsilon_\perp \varepsilon_\parallel}{\varepsilon_\perp \sin^2 \theta_2'' + \varepsilon_\parallel \cos^2 \theta_2''} \right)^{1/2}$$

Here $k_0 = \omega/c$ and θ_2'' is the angle between the wave vector in the crystal, $\mathbf{k}_2$, and the optic axis. Whence, for the ordi-

nary ray

$$\tan\theta_2'' = \frac{\sqrt{\varepsilon_{\|}}\sin\theta_1}{\sqrt{\varepsilon_{\perp}(\varepsilon_{\|}-\sin^2\theta_1)}}$$

The direction of propagation of the extraordinary ray θ_2'' is connected with Θ_2'' by the relationship

$$\tan\Theta_2'' = \frac{\varepsilon_{\perp}}{\varepsilon_{\|}}\tan\theta_2''$$

and so

$$\tan\Theta_2'' = \frac{\sqrt{\varepsilon_{\perp}}\sin\theta_1}{\sqrt{\varepsilon_{\|}(\varepsilon_{\|}-\sin^2\theta_1)}}.$$

89. An electron in a variable electric field $E_0e^{-i\omega t}$ moves according to the following equations:

$$m\ddot{x} = -k_x x + eE_{0x}e^{-i\omega t}$$

$$m\ddot{y} = -k_y y + eE_{0y}e^{-i\omega t}$$

$$m\ddot{z} = -k_z z + eE_{0z}e^{-i\omega t}$$

We seek the solution in the form

$$x = x_0e^{-i\omega t},\quad y = y_0e^{-i\omega t},\quad z = z_0e^{-i\omega t}$$

where x_0, y_0, and z_0 can be found from the equations

$$x_0 = \frac{eE_{0x}}{k_x - m\omega^2},\quad y_0 = \frac{eE_{0y}}{k_y - m\omega^2},\quad z_0 = \frac{eE_{0z}}{k_z - m\omega^2}$$

The projections of the polarization vector induced by the electric field are

$$P_x = \frac{Ne^2}{k_x - m\omega^2}E_x,\quad P_y = \frac{Ne^2}{k_y - m\omega^2}E_y,\quad P_z = \frac{Ne^2}{k_z - m\omega^2}E_z$$

The components of the permittivity tensor are

$$\varepsilon_{xx} = 1 + \frac{Ne^2}{\varepsilon_0(k_x - m\omega^2)},\quad \varepsilon_{yy} = 1 + \frac{Ne^2}{\varepsilon_0(k_y - m\omega^2)}$$

$$\varepsilon_{zz} = 1 + \frac{Ne^2}{\varepsilon_0(k_z - m\omega^2)},\quad \varepsilon_{xy} = \varepsilon_{xz} = \varepsilon_{yz} = 0.$$

92. Choose the z-axis in the direction of the magnetic field. An electron in a variable electric field and a constant

magnetic field moves according to the following equations:

$$\ddot{x}+\omega_0^2 x=\frac{e}{m}E_x+\frac{e}{m}\dot{y}B_0$$

$$\ddot{y}+\omega_0^2 y=\frac{e}{m}E_y-\frac{e}{m}\dot{x}B_0$$

where ω_0 is the natural frequency of oscillations of the electron. Introducing new variables

$$r_+=x+iy,\qquad r_-=x-iy,$$
$$E_+=E_x+iE_y,\quad E_-=E_x-iE_y$$

and considering that the light wave is monochromatic ($E_\pm \propto e^{i\omega t}$), we get

$$r_+=\frac{e}{m}\,\frac{E_+}{\omega_0^2-\omega^2-\frac{e}{m}\omega B_0},\quad r_-=\frac{e}{m}\,\frac{E_-}{\omega_0^2-\omega^2+\frac{e}{m}\omega B_0}$$

Maxwell's equations give the solution for E_+ and E_- in the form of plane waves:

$$E_\pm=E_{0\pm}e^{-i\omega t+ik_\pm z}$$

where

$$k_\pm=\frac{\omega}{c}n_\pm,\quad n_\pm=1+\frac{Ne^2}{\varepsilon_0 m}\,\frac{1}{\omega_0^2-\omega^2\mp\frac{e}{m}\omega B_0}$$

Assume that when $z=0$, the electric field vector in the medium is directed along the y-axis and in magnitude is equal to E_0. Then

$$E_x=E_0 e^{-i\omega t+i\frac{k_++k_-}{2}z}\cos\frac{k_--k_+}{2}z$$

$$E_y=E_0 e^{-i\omega t+i\frac{k_++k_-}{2}z}\sin\frac{k_--k_+}{2}z$$

From this solution we see that vector **E** rotates. Along a path of length l the vector rotates through an angle $\varphi=\frac{k_--k_+}{2}l$.

93. The electric field vector in a monochromatic wave is

$$\mathbf{E}=\mathbf{E}_0e^{i\omega t}+\mathbf{E}_0e^{-i\omega t}\qquad(1)$$

The law of motion of an anharmonic oscillator subjected to an electric field is

$$m\frac{d^2x_i}{dt^2}+m\omega_0^2x_i+\sum_{j,\,l}\beta_{ijl}x_jx_l=eE_i \tag{2}$$

Assuming β_{ijl} to be small, we expand x_i in a series

$$x_i=x_i^{(0)}(t)+x_i^{(1)}(t)+\ldots$$

where $x_i^{(1)}$ is the first-order term in the expansion of x_i in powers of β_{ijl}.

For the zero approximation we exclude the terms with β_{ijl} in Eq. (2) and get

$$x_i^{(0)}=\frac{eE_i}{m(\omega_0^2-\omega^2)}$$

In the next approximation

$$x_i^{(1)}=-\frac{1}{2}\sum_{j,\,l}\frac{e^2\beta_{ijl}E_{0j}E_{0l}}{m^3(\omega_0^2-\omega^2)^2}\left[\frac{1}{\omega_0^2}+\frac{\cos 2\omega t}{\omega_0^2-4\omega^2}\right]$$

The polarization vector is

$$\mathbf{P}=\mathbf{P}^{(0)}+\mathbf{P}^{(1)}$$

where

$$\mathbf{P}^{(0)}=\frac{Ne^2E}{m(\omega_0^2-\omega^2)}$$

$$\mathbf{P}^{(1)}=\mathbf{P}^{(1)}(0)+\mathbf{P}^{(1)}(2\omega)$$

$$P_i^{(1)}(0)=-\frac{Ne^3}{2}\sum_{j,\,l}\frac{\beta_{ijl}}{m^3}\,\frac{E_{0j}E_{0l}}{(\omega_0^2-\omega^2)^2\,\omega_0^2},\ i=x,\ y,\ z$$

$$P_i^{(1)}(2\omega)=-\frac{Ne^3}{2}\sum_{j,\,l}\frac{\beta_{ijl}}{m^3}\,\frac{E_{0j}E_{0l}\cos 2\omega t}{(\omega_0^2-4\omega^2)(\omega_0^2-\omega^2)^2}.$$

94. $\mathbf{P}^{(1)}=\mathbf{P}^{(1)}(\omega_1-\omega_1)+\mathbf{P}^{(1)}(2\omega_1)+\mathbf{P}^{(1)}(\omega_2-\omega_2)+$

$+\mathbf{P}^{(1)}(2\omega_2)+2\mathbf{P}^{(1)}(\omega_1+\omega_2)+2\mathbf{P}^{(1)}(\omega_1-\omega_2).$

The first four members are just $\mathbf{P}^{(1)}(0)$ and $\mathbf{P}^{(1)}(2\omega)$ of the previous problem, and the projections of $\mathbf{P}^{(1)}(\omega_1\pm\omega_2)$

are

$$P_i^{(1)}(\omega_1 \pm \omega_2) = -\frac{Ne^3}{2} \sum_{j,\,l} \frac{\beta_{ijl}}{m^3} \frac{E_{0j}E_{0l}\cos(\omega_1 \pm \omega_2)t}{\omega_0^2 - (\omega_1 \pm \omega_2)^2} \times \frac{1}{(\omega_0^2 - \omega_1^2)(\omega_0^2 - \omega_2^2)}.$$

95. Let the z-axis be directed along the length of the chains in the direction of propagation of the wave. The nth oscillator in a chain has the following law of motion

$$m\ddot{\mathbf{r}}_n + m\omega_0^2\mathbf{r}_n - q[(\mathbf{r}_{n+1} - \mathbf{r}_n) + (\mathbf{r}_{n-1} - \mathbf{r}_n)] = e\mathbf{E}_n$$

where q is the elastic constant.

Multiply this equation by e/v_0, where v_0 is the volume per oscillator. The quantity $e\mathbf{r}_n/v_0 = \mathbf{P}_n$ is the polarization vector. The law of motion then takes the form

$$m\ddot{\mathbf{P}}_n + m\omega_0^2\mathbf{P}_n - q[(\mathbf{P}_{n+1} - \mathbf{P}_n) + (\mathbf{P}_{n-1} - \mathbf{P}_n)] = \frac{e^2\mathbf{E}_n}{v_0}$$

If the wavelength is considerably greater than the chain's period, the polarization vector is a slowly varying function. Then the equation can be rewritten in the form

$$m\ddot{\mathbf{P}}(z) + m\omega_0^2\mathbf{P}(z) - qa^2\frac{d^2\mathbf{P}(z)}{dz^2} = \frac{e^2}{v_0}\mathbf{E}(z) \qquad (1)$$

where a is the period.

This equation must be solved together with the equation for the electric field

$$\frac{\partial^2\mathbf{E}}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2} = \frac{\partial^2\mathbf{P}}{\varepsilon_0 c^2\,\partial t^2} \qquad (2)$$

We seek the solution of (1) and (2) in the form

$$\mathbf{P} = \mathbf{P}_0 e^{ikz - i\omega t}, \quad \mathbf{E} = \mathbf{E}_0 e^{ikz - i\omega t} \qquad (3)$$

Substituting (3) into (1) and (2), we get

$$\frac{\omega^2}{c^2} - k^2 = -\frac{\alpha}{\omega_0^2 - \omega^2 + \frac{q}{m}k^2a^2}, \quad \text{where} \quad \alpha = \frac{\omega^2 e^2}{\varepsilon_0 mc^2 v_0}$$

Whence

$$k = \pm\left(\frac{q\frac{\omega^2a^2}{c^2m} - \omega_0^2 + \omega^2 \pm \sqrt{\left(q\frac{\omega^2a^2}{c^2m} - \omega_0^2 + \omega^2\right)^2 + 4\alpha\frac{qa^2}{m}}}{2qa^2/m}\right)^{1/2}$$

Thus for a given frequency there are four wave numbers $\left(\text{and, consequently, four indexes of refraction, which are connected with each } k \text{ by a relationship } n = \frac{c}{\omega} k\right)$. Two waves move in one direction; two others, with the same indexes of refraction as the first two, move in the opposite direction. In an isotropic medium it is possible for waves with different indexes of refraction and the same frequency to move in one direction. This property is inherent in media with so-called spatial dispersion, i.e. when permittivity depends not only on the frequency of the wave but on the wave vector as well.

96. Direct the x-axis perpendicular to the plates, and let the planes where the plates lie be $x = 0$ and $x = d$. The z-axis is then directed along the line of propagation of the wave. We seek the solution of Maxwell's equations in the form

$$\left.\begin{aligned} \mathbf{E}(\mathbf{r}, t) &= \mathbf{E}(x)\, e^{ikz-i\omega t} \\ \mathbf{H}(\mathbf{r}, t) &= \mathbf{H}(x)\, e^{ikz-i\omega t} \end{aligned}\right\} \tag{1}$$

Substitution in Maxwell's equations yields the following system of equations:

$$\left.\begin{aligned} kH_y &= \omega\varepsilon\varepsilon_0 E_x, & -kE_y &= \omega\mu_0 H_x \\ kH_x - \frac{\partial H_z}{\partial x} &= -i\omega\varepsilon\varepsilon_0 E_y, & ikE_x - \frac{\partial E_z}{\partial x} &= i\omega\mu_0 H_y \\ \frac{\partial H_y}{\partial x} &= -i\omega\varepsilon\varepsilon_0 E_z, & \frac{\partial E_y}{\partial x} &= i\omega\mu_0 H_z \\ \frac{\partial H_x}{\partial x} + ikH_z &= 0, & \frac{\partial E_x}{\partial x} + ikE_z &= 0 \end{aligned}\right\} \tag{2}$$

The boundary conditions for our case of ideally conducting plates are

$$E_y = E_z = H_x = 0 \quad \text{when} \quad x = 0 \quad \text{and} \quad x = d$$

The system of equations (2) separates into two independent systems; one for H_x, H_z and E_y, and the other for E_x, E_z and H_y. The first describes the propagation of TE waves, and the second of TM waves.

For TE waves ($E_z = 0$),

$$\frac{\partial^2 H_z}{\partial x^2} + \varkappa^2 H_z = 0 \tag{3}$$

$$E_y = -i\frac{\omega\mu_0}{\varkappa^2}\frac{\partial H_z}{\partial x}, \quad H_x = \frac{ik}{\varkappa^2}\frac{\partial H_z}{\partial x} \tag{4}$$

where

$$\varkappa^2 = \frac{\omega^2 \varepsilon}{c^2} - k^2 \tag{5}$$

The boundary conditions are: $\frac{\partial H_z}{\partial x} = 0$ for $x = 0$ and $x = d$. The general solution of (3) is of the form

$$H_z = H_{01} \sin \varkappa x + H_{02} \cos \varkappa x$$

where the boundary conditions impose the restriction that

$$H_{02} = 0; \quad \varkappa = \pi l/d, \quad l = 1, 2, \ldots$$

Thus for TE waves,

$$H_z(x) = H_{01} \sin \frac{\pi l}{d} x \tag{6}$$

For each value of l there is a definite type of wave that can propagate in the system. The magnitudes of the fields for each type can be found from Eqs. (1), (4) and (6), and the dispersion equation is simply formula (5).

For TM waves ($H_z = 0$),

$$\frac{\partial^2 E_z}{\partial x^2} + \varkappa^2 E_z = 0 \tag{7}$$

$$H_y = i\frac{\omega\varepsilon\varepsilon_0}{\varkappa^2}\frac{\partial E_z}{\partial x}, \quad E_x = \frac{ik}{\varkappa^2}\frac{\partial E_z}{\partial x}$$

The boundary conditions are: $E_z = 0$ for $x = 0$ and $x = d$. The solution of Eq. (7) together with the boundary conditions is of the form

$$E_z(x) = E_0 \cos \frac{\pi l}{d} x, \quad l = 0, 1, 2, \ldots .$$

97. Direct the z-axis along the wave guide. For a travelling wave we then have $E \propto e^{ikz}$, $H \propto e^{ikz}$. The magnetic field component along the line of propagation of the TE wave

is determined from the equation

$$\frac{\partial^2 H_z}{\partial x^2}+\frac{\partial^2 H_z}{\partial y^2}+\gamma^2 H_z=0$$

where $\gamma^2=\frac{\omega^2}{c^2}-k^2$. The boundary conditions are: $\frac{\partial H_z}{\partial n}=0$ when $x=0$, $x=a$, $y=0$, and $y=b$.

The solution of the equation is of the form

$$H_{zmn}(x, y)=H_0\cos\frac{\pi mx}{a}\cos\frac{\pi ny}{b}$$

where m and n are integers;

$$\gamma^2\to\gamma_{mn}^2=\pi^2\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)$$

For $a>b$ the frequency limit is $\omega_{01}=\pi c/a$, for $a<b$ it is $\omega_{01}=\pi c/b$.

For TM waves

$$E_{zmn}(x, y)=E_0\sin\frac{\pi mx}{a}\sin\frac{\pi ny}{b}.$$

98. The electromagnetic field in a conductor decreases sharply at depths below the surface considerably less than the wavelength in a vacuum. This is the reason why the wave vector is directed perpendicular to the surface. The electric and magnetic fields in the conductor are related to each other by the equality

$$\sqrt{\varepsilon\varepsilon_0+\frac{\sigma^* i}{\omega}}\,\mathbf{E}_t=\sqrt{\mu\mu_0}\,[\mathbf{H}_t\times\mathbf{n}]$$

where $\mathbf{E}_t$ and $\mathbf{H}_t$ are the tangential components of the fields, and $\mathbf{n}$ is the normal vector. Since $\mathbf{E}_t$ and $\mathbf{H}_t$ are continuous, we have the same relation outside the conductor. For an ideal conductor,

$$\mathbf{E}_t=\zeta\,[\mathbf{H}_t\times\mathbf{n}],\quad \text{where}\quad \zeta=\sqrt{\mu\omega\mu_0/(\sigma^* i)}.$$

99. Use the conditions of Problem 98. The electric and magnetic fields decrease like $e^{-\alpha z}$, where $\alpha=2\varepsilon_0\left(\frac{\omega\mu}{2\sigma^*}\right)^{1/2}\times$

$$\times\,\pi^2\left(b\frac{m^2}{a^2}+a\frac{n^2}{b^2}\right)\left[\gamma_{mn}^2\left(\frac{\omega^2}{c^2}-\gamma_{mn}^2\right)^{1/2}\right]^{-1}.$$

100. For TM waves,

$$E_{zn}=E_0 J_n(\gamma r)\begin{cases}\sin n\varphi\\ \cos n\varphi\end{cases}$$

where J_n is the nth Bessel function (n an integer), and γ is determined from the condition that $J_n(\gamma R)=0$.

For TE waves,

$$H_{zn}=H_0 J_n(\gamma r)\begin{cases}\sin n\varphi\\ \cos n\varphi\end{cases}$$

where γ is determined from the condition $J_n'(\gamma R)=0$.

The dispersion equation for TE and TM waves is

$$k=\left(\frac{\omega^2}{c^2}-\gamma^2\right)^{1/2}.$$

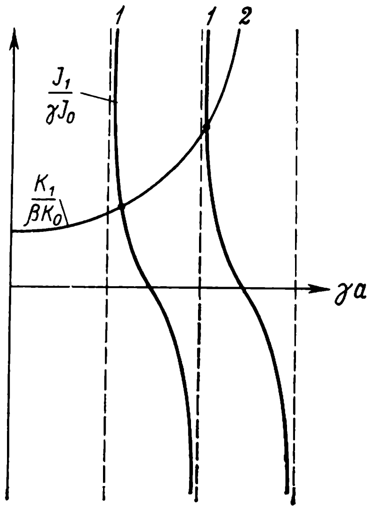

Fig. 57

101. Solving Maxwell's equations inside and outside the guide and fitting both solutions on the boundary, we come to the conditions

$$\frac{J_1(\gamma R)}{\gamma J_0(\gamma R)}+\frac{K_1(\beta R)}{\beta K_0(\beta R)}=0 \qquad (1)$$

$$\gamma^2+\beta^2=(\varepsilon-1)\frac{\omega^2}{c^2} \qquad (2)$$

where $\frac{\varepsilon\omega^2}{c^2}-k^2=\gamma^2$, $\frac{\omega^2}{c^2}-k^2=-\beta^2$, and K_0 and K_1 are modified Hankel functions.

The graphical solution of Eqs. (1) and (2) is given in Fig. 57. For frequencies less than a definite value, called the frequency limit, curves of (1) and (2) do not intersect and β is imaginary. In this case the waves do not propagate along the guide but are radiated in the surrounding medium.

102. Direct the z-axis along the common axis of the cylinders. For a wave propagating along this axis ($E, H \propto e^{ikz-i\omega t}$) Maxwell's equations in cylindrical coordinates

are

$$\left.\begin{aligned}
&\frac{1}{r}\frac{\partial H_z}{\partial \varphi} - ikH_\varphi = -i\omega\varepsilon\varepsilon_0 E_r\\
&ikH_r - \frac{\partial H_z}{\partial r} = -i\omega\varepsilon\varepsilon_0 E_\varphi\\
&\frac{1}{r}\frac{\partial}{\partial r}(rH_\varphi) - \frac{1}{r}\frac{\partial H_r}{\partial \varphi} = -i\omega\varepsilon\varepsilon_0 E_z\\
&\frac{1}{r}\frac{\partial E_z}{\partial \varphi} - ikE_\varphi = i\omega\mu_0 H_r\\
&ikE_r - \frac{\partial E_z}{\partial r} = i\omega\mu_0 H_\varphi\\
&\frac{1}{r}\frac{\partial}{\partial r}(rE_\varphi) - \frac{1}{r}\frac{\partial E_r}{\partial \varphi} = i\omega\mu_0 H_z
\end{aligned}\right\} \tag{1}$$

Also $E_\varphi = H_z = 0$ for $r = R_1$ or $r = R_2$. Let us show that for a wave guide with a multiply connected cross section (which is what our system is) there is a possibility of transverse waves with $E_z = H_z = 0$, $k = \frac{\omega}{c}\sqrt{\varepsilon}$. For this case it follows from the system of equations (1) that

$$\left.\begin{aligned}
&\sqrt{\varepsilon}\, E_r = H_\varphi \sqrt{\mu_0/\varepsilon_0}\\
&\sqrt{\varepsilon}\, E_\varphi = H_r \sqrt{\mu_0/\varepsilon_0}\\
&\frac{\partial}{\partial r}(rE_\varphi) - \frac{\partial E_r}{\partial \varphi} = 0\\
&\frac{\partial}{\partial r}(rH_\varphi) - \frac{\partial H_r}{\partial \varphi} = 0
\end{aligned}\right\} \tag{2}$$

Using these equations, we can show that rE_r, rE_φ, rH_r, and rH_φ must satisfy the two-dimensional Laplace equation

$$\frac{\partial}{\partial r}\left(r\frac{\partial U}{\partial r}\right) + \frac{1}{r}\frac{\partial^2 U}{\partial \varphi^2} = 0 \tag{3}$$

Since rE_φ and rH_r are zero on the boundary of the region and are harmonic functions inside, this implies that $E_\varphi = H_r = 0$ everywhere. Then (2) yields

$$\frac{\partial E_r}{\partial \varphi} = 0, \quad \frac{\partial (rH_\varphi)}{\partial r} = 0$$

Whence, having in mind the dependence on z and t, we get

$$H_\varphi = \frac{A}{r} e^{ikz-i\omega t}, \quad E_r = \sqrt{\frac{\mu_0}{\varepsilon_0 \varepsilon}} \frac{A}{r} e^{ikz-i\omega t}$$

For the TM waves we must solve an equation that follows from Eq. (1):

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial E_z}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 E_z}{\partial \varphi^2} + \varkappa^2 E_z = 0 \tag{4}$$

The solution is of the form

$$E_{nz} = [A_n J_n (\varkappa r) + B_n Y_n (\varkappa r)] \begin{cases} \sin n\varphi \\ \cos n\varphi \end{cases} \tag{5}$$

where $J_n (\varkappa r)$ is the nth Bessel function of the first kind, and $Y_n (\varkappa r)$ is the nth Weber's Bessel function of the second kind.

As distinct from wave guides of a circular cross section, we have no need to discard solutions that at $r = 0$ become infinite, since point $r = 0$ does not belong to the system.

The boundary conditions yield

$$\left. \begin{aligned} A_n J_n (\varkappa R_1) + B_n Y_n (\varkappa R_1) &= 0 \\ A_n J_n (\varkappa R_2) + B_n Y_n (\varkappa R_2) &= 0 \end{aligned} \right\} \tag{6}$$

This system of equations has nontrivial solutions if the system determinant is zero, i.e.

$$J_n (\varkappa R_1) Y_n (\varkappa R_2) - J_n (\varkappa R_2) Y_n (\varkappa R_1) = 0 \tag{7}$$

The last equation defines the values of $\varkappa$ for a given n: $\varkappa_{n1}, \varkappa_{n2}, \ldots$. For the TE waves Eq. (7) contains J_n' and Y_n' instead of J_n and Y_n.

103. $E_x = E_{x0} \cos \alpha x \sin \beta y \sin \gamma z e^{-i\omega t}$

$E_y = E_{y0} \sin \alpha x \cos \beta y \sin \gamma z e^{-i\omega t}$

$E_z = E_{z0} \sin \alpha x \sin \beta y \cos \gamma z e^{-i\omega t}$

where $\alpha = \pi m/a$, $\beta = \pi n/b$, $\gamma = \pi l/c$, and $\omega^2 = c^2\pi^2 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} + \frac{l^2}{c^2} \right)$ [m, n, l integers].

The lowest frequency is

$$\omega_{\min} = c\frac{\pi}{L_i}$$

where L_i is the longer edge of the parallelepiped.

105. From the condition that $E_t = 0$ at the ends of the resonator we come to a solution in the form

$$E_z = Af(\rho, \varphi)\cos kz\, e^{-i\omega t}$$

where $f(\rho, \varphi)$ satisfies the equation

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial f}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial\varphi^2} + \gamma^2 f = 0, \quad \gamma^2 = \frac{\omega^2}{c^2} - k^2 \qquad (1)$$

and $k = \pi n/d$ (n is an integer). Separation of variables brings us to the Bessel equation for the radial component. Thus

$$E_z = E_0 J_m(\gamma_l\rho)\, e^{im\varphi}\cos kz\, e^{-i\omega t}$$

where J_m is the mth Bessel function (m is an integer), and γ_l is determined from the condition that $J_m(\gamma_l R) = 0$; l numbers the different roots of a Bessel function.

The natural frequencies of the TM waves can be found from

$$\frac{\omega^2}{c^2} = \gamma_l^2 + \frac{n^2\pi^2}{d^2}$$

For TE waves the natural frequencies are determined from the condition that $J'_m(\gamma_l R) = 0$.

106. Let the liquid move along the x-axis. The velocity at any point in the liquid depends only on the position of that point between the plates (on coordinate z). So does the magnetic field.

We must now solve the system of equations (II-74) and (II-75) together with the boundary conditions: $v = 0$ and $H_x = 0$ when $z = \pm d/2$. Equation $\operatorname{div} \mathbf{H} = 0$ yields

$$H_z = \text{constant} = H_0$$

For the z-component of Eq. (II-75) we have

$$p + \frac{\mu_0 H_x^2}{2} = P(x)$$

and $\frac{dp}{dx}=\frac{dP}{dx}=\text{constant}$ because of stationary flow. Then Eqs. (II-75)-(II-77) yield

$$H_0\frac{dv}{dz}+\frac{c^2\varepsilon_0}{\sigma^*}\frac{d^2H_x}{dx^2}=0$$

$$\eta\frac{d^2v}{dz^2}+\mu_0H_0\frac{dH_x}{dz}=\frac{dp}{dx}$$

Solutions of these equations that satisfy the boundary conditions have the form

$$v=v_0\frac{\cosh\frac{d}{2d_0}-\cosh\frac{z}{d_0}}{\cosh\frac{d}{2d_0}-1}$$

$$H_x=-v_0 4\pi\varepsilon_0\sqrt{\sigma^*\eta}\,\frac{\frac{2z}{d}\sinh\frac{d}{2d_0}-\sinh\frac{z}{d_0}}{\cosh\frac{d}{2d_0}-1}$$

where $d_0=\frac{1}{H_0}\sqrt{\frac{\eta}{\sigma^*\mu_0}}$ and v_0 is the velocity of the liquid at $z=0$.

For weak fields $(d\ll d_0)$, $v=v_0\left(1-\frac{4z^2}{d^2}\right)$. This is simply the result of ordinary hydrodynamics.

For strong fields $(d\gg d_0)$, $v=v_0\left(1-e^{-\left(\frac{d}{2}-|z|\right)\frac{1}{d_0}}\right)$.

107. Equation (II-75) for our case is

$$\frac{\partial p}{\partial x}=\mu_0\sigma H_0(E_0-\mu_0H_0v)+\eta\frac{d^2v}{dz^2} \tag{1}$$

$$\frac{\partial p}{\partial y}=0 \tag{2}$$

$$\frac{\partial p}{\partial z}=-\mu_0\sigma H_x(E_0-\mu_0H_0v) \tag{3}$$

Since there is no pressure gradient along the x-axis, we come to the equation for the distribution of velocities:

$$\frac{d^2v}{dz^2}-\frac{1}{d_0^2}v=\frac{1}{d_0^2}\frac{E_0}{\mu_0H_0}$$

Solving this equation and using the boundary conditions ($v=0$ when $z=0$, and $v=v_0$ when $z=d$), we come to

the sought distribution

$$v(z)=\frac{v_0}{\sinh\frac{d}{d_0}}\sinh\frac{z}{d_0}+\frac{E_0}{\mu_0 H_0}\left[1-\frac{\sinh\frac{d-z}{d_0}+\sinh\frac{z}{d_0}}{\sinh\frac{d}{d_0}}\right]$$

From Eq. (3) we obtain

$$H_x(z)=H_0v_0\sigma dd_0\frac{\cosh\frac{d}{2d_0}-\cosh\left(\frac{d}{2d_0}-\frac{z}{d_0}\right)}{2\sinh\frac{d}{2d_0}}$$

For weak fields ($d\ll d_0$),

$$v=\frac{v_0z}{d}$$

For strong fields ($d\gg d_0$),

$$v=\frac{E_0}{\mu_0H_0}(1-e^{-z/d_0}).$$

108. We write the continuity equation in terms of the concentration of the electrons:

$$\frac{\partial n}{\partial t}+\operatorname{div} n\mathbf{v}=0 \tag{1}$$

The law of motion of an electron in a plasma with no pressure gradient is

$$\frac{d\mathbf{v}}{dt}=\frac{\partial\mathbf{v}}{\partial t}+(\mathbf{v}\cdot\nabla)\,\mathbf{v}=\frac{e}{m}\mathbf{E} \tag{2}$$

Maxwell's equations yield

$$\operatorname{div}\mathbf{E}=\frac{e}{\varepsilon_0}(n-n_0) \tag{3}$$

where n_0 is the ion concentration and also the mean electron concentration.

Let us assume the motion of charges to be so slow that the variation of charge density is considerably less than its mean value. In such an approximation Eqs. (1)-(3) trans-

form into the following:

$$\frac{\partial \Delta n}{\partial t}+n_0 \operatorname{div} \mathbf{v}=0$$

$$\operatorname{div} \mathbf{E}=\frac{e}{\varepsilon_0} \Delta n$$

$$\frac{\partial \mathbf{v}}{\partial t}=\frac{e}{m} \mathbf{E}$$

We then find the divergence of both sides of the last equation. Excluding div **v** and div **E** with the help of the first two equations, we find

$$\frac{\partial^2 \Delta n}{\partial t^2}+\frac{e^2}{m\varepsilon_0} n_0 \Delta n=0$$

The solution of this equation is

$$\Delta n=\Delta n_0 e^{-i\omega_p t}$$

where $\omega_p=\sqrt{\frac{e^2 n_0}{\varepsilon_0 m}}$ is the plasma frequency.

109. $\Delta\varphi=-\frac{\rho}{\varepsilon_0}$ and $\Delta \mathbf{A}-\frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2}=-\mu_0 \mathbf{j}_\perp$, where $\mathbf{j}_\perp$ is the transverse component of the current density vector:

$$\mathbf{j}_\perp=\frac{1}{4\pi} \operatorname{curl} \operatorname{curl} \int \frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} dV'.$$

110. The vector potential is equal to

$$\mathbf{A}(\mathbf{r})=\frac{\mu_0}{4\pi} \int \frac{\mathbf{j}(\mathbf{r}') e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} dV'$$

When the distances from the antenna are great, the vector potential is

$$\mathbf{A}(\mathbf{r})=\mathbf{e}_3 I \frac{e^{ikr}\mu_0}{4\pi r} \int\limits_{-d/2}^{d/2} \sin\left(\frac{kd}{2}-k|z'|\right) e^{-ikz'\cos\theta} dz'$$

where θ is the angle between $\mathbf{e}_3$ and the direction of propagation of waves. Evaluating the integral, we get

$$\mathbf{A}(\mathbf{r})=\mathbf{e}_3 \frac{\mu_0 I e^{ikr}}{2\pi r} \frac{\cos\left(\frac{kd}{2}\cos\theta\right)-\cos\frac{kd}{2}}{\sin^2\theta}$$

$$H=\frac{\omega}{c\mu_0} A_3 \sin\theta$$

The time average radiation intensity per unit of solid angle is

$$\frac{dJ}{d\Omega}=\frac{1}{2}\operatorname{Re}\left(r^2\mathbf{n}\cdot[\mathbf{E}\times\mathbf{H}^*]\right)$$

$$=\frac{I^2}{8\pi^2 c\varepsilon_0}\left|\frac{\cos\left(\frac{kd}{2}\cos\theta\right)-\cos\frac{kd}{2}}{\sin\theta}\right|^2$$

When the antenna length is several half-lengths,

$$\frac{dJ}{d\Omega}=\sqrt{\frac{\mu_0}{\varepsilon_0}}\,\frac{I^2}{8\pi^2}\begin{cases}\dfrac{\cos^2\left(\frac{\pi}{2}\cos\theta\right)}{\sin^2\theta} & (kd=\pi)\\[2ex] \dfrac{4\cos^4\left(\frac{\pi}{2}\cos\theta\right)}{\sin^2\theta} & (kd=2\pi).\end{cases}$$

111. $$J=\frac{I_0^2}{4\pi}\sqrt{\frac{\mu_0}{\varepsilon_0}}\left[\int_0^{2kl}\frac{1-\cos u}{u}\,du-\frac{1}{2kl}\int_0^{2kl}(1-\cos u)\,du\right].$$

113. The magnetic field of the incident wave in cylindrical coordinates can be written as follows:

$$H_z=H_{0z}e^{i(k\rho\cos\varphi-\omega t)}$$

The plane wave can be expanded into a Bessel series:

$$H_z=H_{0z}\sum_{m=-\infty}^{\infty}J_m(k\rho)\,i^m e^{i(m\varphi-\omega t)}$$

The symmetry of the problem implies that the magnetic field of the scattered wave will also be directed along the cylinder's axis. Let us look for this field in the form of a series of cylindrical functions whose asymptotic behaviour for large ρ is the same as for the field

$$H\sim\rho^{-1/2}e^{ik\rho}$$

These are the Hankel functions $H_m(k\rho)$. For the scattered wave we then have

$$H_z=H_{0z}\sum_{m=-\infty}^{\infty}a_m H_m(k\rho)\,e^{i(m\varphi-\omega t)}$$

The boundary condition is: $H_z = 0$ on the surface of the cylinder. For our problem this is equivalent to $E_\varphi = 0$. The equation curl $\mathbf{H} = \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$ leads us to the expression

$$E_\varphi = -\frac{i}{\varepsilon_0 \omega} \frac{\partial H_z}{\partial \rho}$$

The boundary conditions yield

$$a_m = (-i)^m \frac{J'_m(kR)}{H'_m(kR)}$$

For large ρ,

$$H_m(k\rho) \sim \sqrt{\frac{2}{\pi k \rho}}\, e^{ik\rho - (2m+1)\frac{\pi}{4} i}$$

Then we find the magnetic field for large ρ:

$$H_z = H_{0z} \sqrt{\frac{2}{\pi k \rho}}\, e^{i(k\rho - \omega t)} \sum_{m=-\infty}^{\infty} a_m e^{i\left(m\varphi - \frac{2m+1}{4}\pi\right)}$$

$$E_\varphi = \sqrt{\frac{\mu_0}{\varepsilon_0}}\, H_z$$

The time average of the energy flux density in the scattered wave is

$$|\mathbf{S}| = \frac{1}{2} \operatorname{Re} (E_\varphi H_z^*) = \frac{1}{2} \sqrt{\frac{\mu_0}{\varepsilon_0}}\, |H_z^*|^2$$

The energy flux through a unit lengtn of the cylinder is

$$J = \int_0^{2\pi} |\mathbf{S}|\, \rho\, d\varphi = 2 \sqrt{\frac{\mu_0}{\varepsilon_0}}\, |H_{0z}|^2 \times \frac{1}{k} \sum_{m=-\infty}^{\infty} \left| \frac{J'_m(kR)}{H'_m(kR)} \right|^2.$$

114. The dipole moment of a system of two particles in their centre-of-mass system is

$$\mathbf{p} = e_1 \mathbf{r}_1 + e_2 \mathbf{r}_2 = \mu \left(\frac{e_1}{m_1} - \frac{e_2}{m_2} \right) \mathbf{r}$$

where m_1 and m_2 are the masses of the particles, μ is their reduced mass, and $\mathbf{r}$ is their separation.

For identical particles, $e_1 = e_2$ and $m_1 = m_2$. And so $\mathbf{p} = 0$, which means that there is no dipole radiation, i.e. radiation proportional to $|\mathbf{p}|^2$.

115. The intensity of dipole radiation can be calculated using formula (II-73), where $\mathbf{p} = e\mathbf{r}$. With the help of the law of motion

$$m\ddot{\mathbf{r}} = -\frac{e^2\mathbf{r}}{4\pi\varepsilon_0 r^3}$$

we can exclude $\ddot{r}$. Whence,

$$J = \frac{e^6}{96\pi^3\varepsilon_0^3 c^3 a^4 m^2} = \frac{128\pi\varepsilon_0 \mid E \mid^4}{3m^2c^3e^2}$$

where E is the particle's energy.

116. If in the course of one period of revolution the particle's energy changes only slightly, we can use the result of Problem 115 and find that

$$\frac{dE}{dt} = \frac{128\pi\varepsilon_0 \mid E \mid^4}{3m^2c^3e^2}$$

whence

$$\tau = \frac{3m^2c^3e^2}{4\pi\varepsilon_0 \times 24 \mid E \mid^3}.$$

117. The time dependence of the electromagnetic field can be presented in the form

$$E(t) = \begin{cases} 0 & \text{for} \quad t < 0 \\ E_0 e^{-\frac{t}{\tau} - i\omega_0 t} & \text{for} \quad t > 0 \end{cases}$$

Let us write $E(t)$ in the form of a Fourier integral

$$E(t) = \int_{-\infty}^{\infty} E(\omega)\, e^{-i\omega t}\, d\omega$$

where the Fourier transform

$$E(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} E(t)\, e^{+i\omega t}\, dt = \frac{E_0}{2\pi\left[\frac{1}{\tau} - i(\omega_0 - \omega)\right]}$$

The radiation intensity is proportional to the squared modulus of the Fourier transform:

$$J(\omega) = \frac{J_0}{\pi\tau\left[\frac{1}{\tau^2} + (\omega_0 - \omega)^2\right]}$$

where

$$J_0 = \int_{-\infty}^{\infty} J(\omega)\, d\omega$$

The half-width of the radiated line is determined from the relationship

$$J\left(\omega_0 \pm \frac{\Delta\omega}{2}\right) = \frac{1}{2} J(\omega_0)$$

whence

$$\Delta\omega = \frac{2}{\tau}.$$

118. We can represent an elliptically polarized wave in the form

$$\mathbf{E} = \mathbf{A} \cos \omega t + \mathbf{B} \sin \omega t$$

where $\mathbf{A} \perp \mathbf{B}$. Then the differential cross section is

$$d\sigma = \left(\frac{e^2}{4\pi\varepsilon_0 mc^2}\right)^2 \frac{[\mathbf{A}\times\mathbf{n}]^2 + [\mathbf{B}\times\mathbf{n}]^2}{[(\omega_0^2-\omega^2)^2+\omega^2\gamma^2]\,(A^2+B^2)}\,\omega^4\, d\Omega$$

where $\mathbf{n}$ is a unit vector in the direction of scatter.

119. Let us seek the electric and magnetic fields in the form

$$\left.\begin{aligned} \mathbf{E} &= \mathbf{E}_{01} e^{iq_x x + iq_y y - ik_z z - i\omega t} \\ \mathbf{H} &= \mathbf{H}_{01} e^{iq_x x + iq_y y - ik_z z - i\omega t} \end{aligned}\right\} \quad \text{for} \quad z < 0$$

and

$$\left.\begin{aligned} \mathbf{E} &= \mathbf{E}_{02} e^{iq_x x + iq_y y + ik_z z - i\omega t} \\ \mathbf{H} &= \mathbf{H}_{02} e^{iq_x x + iq_y y + ik_z z - i\omega t} \end{aligned}\right\} \quad \text{for} \quad z > 0$$

where $k_z = \sqrt{\frac{\omega^2}{c^2} - q_x^2 - q_y^2}$.

Choose the y-axis directed along $\mathbf{i}$. Using the boundary conditions (II-25) and (II-26) and also the relation between $\mathbf{E}$

and **H** in a plane wave, we obtain

$$H_{01x}=-\frac{i}{2},\quad H_{01y}=0,\quad H_{01z}=-\frac{q_x}{2k_z},$$

$$E_{01x}=\frac{q_x q_y}{2\varepsilon_0\omega k_z}i,\quad E_{01y}=-\frac{k_z^2+q_x^2}{2\varepsilon_0\omega k_z}i,\quad E_{01z}=-\frac{q_y}{2\varepsilon_0\omega}i,$$

$$H_{02x}=\frac{i}{2},\quad H_{02y}=0,\quad H_{02z}=-\frac{q_x}{2k_z}i,$$

$$E_{02x}=\frac{q_x q_y}{2\varepsilon_0\omega k_z}i,\quad E_{02y}=-\frac{k_z^2+q_x^2}{2\varepsilon_0\omega k_z}i,\quad E_{02z}=\frac{q_y}{2\varepsilon_0\omega}i$$

Electromagnetic waves are radiated in the ambient space if $\omega^2/c^2 > q_x^2+q_y^2$. When this condition is not fulfilled, an electromagnetic field exists only near the plane together with the current.

120. The electric field of the vibrating dipole induces a dipole moment in the particle. The particle then radiates electromagnetic waves. Provided $d \ll \lambda$, the electric field vector at a point with a radius vector **d** is

$$\mathbf{E}_1(\mathbf{d})=\frac{3(\mathbf{p}\cdot\mathbf{d})\,\mathbf{d}}{4\pi\varepsilon_0 d^5}-\frac{\mathbf{p}}{4\pi\varepsilon_0 d^3}$$

When $\mathbf{p}\perp\mathbf{d}$,

$$\mathbf{E}_1(d)=-\frac{\mathbf{p}}{4\pi\varepsilon_0 d^3}$$

This field induces in the particle a moment

$$\mathbf{p}_1=\varepsilon_0\beta\mathbf{E}_1$$

The net field is created by both dipole moments, **p** and $\mathbf{p}_1$. At great distances from the system ($r\gg\lambda$),

$$\mathbf{E}=\mathbf{E}_1+\mathbf{E}_2,\quad \mathbf{H}=\mathbf{H}_1+\mathbf{H}_2$$

where

$$\mathbf{E}_1=-\frac{\omega^2[[\mathbf{p}\times\mathbf{n}]\times\mathbf{n}]}{4\pi\varepsilon_0 c^2 r},\qquad \mathbf{H}_1=-\frac{\omega^2}{4\pi cr}[\mathbf{p}\times\mathbf{n}]$$

$$\mathbf{E}_2=\frac{\omega^2\beta}{(4\pi)^2c^2\varepsilon_0 rd^3}[[\mathbf{p}\times\mathbf{n}]\times\mathbf{n}],\quad \mathbf{H}_2=\frac{\omega^2\beta}{(4\pi)^2 crd^3}[\mathbf{p}\times\mathbf{n}]$$

where $\mathbf{n}=\mathbf{r}/r$.

The radiation intensity of the electromagnetic waves is

$$J=\frac{p^2\omega^4}{6\pi\varepsilon_0 c^3}\left(1-\frac{\beta}{4\pi d^3}\right)^2.$$

121. One of the ways in which the damping manifests itself is the following. As a consequence of the finiteness of the electric conductivity, there exists a flow of electromagnetic energy through the plane, which leads to a loss in the energy of vibration. This energy flux is maximal in the vicinity of the dipole. Since $d \ll \lambda$, we can use the quasi-stationary approximation to find the electromagnetic field

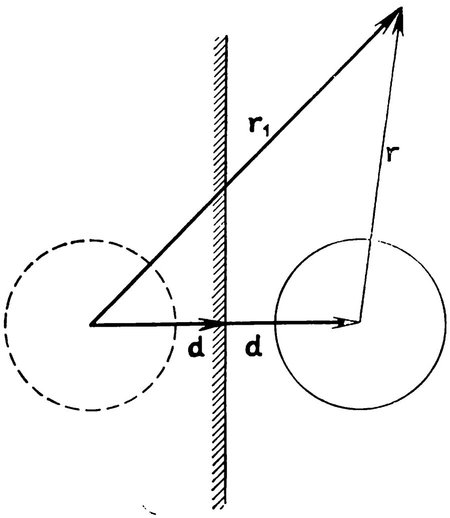

Fig. 58

in the vicinity of the dipole. In this approximation we can show that on the plane the dipole induces electric charges whose field is the field of a dipole that is a mirror image of the original dipole (Fig. 58). And so in the vicinity of the dipole ($r \ll \lambda$),

$$\mathbf{E} = \frac{3(\mathbf{p}\cdot\mathbf{r})\,\mathbf{r}}{4\pi\varepsilon_0 r^5} - \frac{\mathbf{p}}{4\pi\varepsilon_0 r^3} + \frac{3(\mathbf{p}_1\cdot\mathbf{r}_1)\,\mathbf{r}_1}{4\pi\varepsilon_0 r_1^5} - \frac{\mathbf{p}_1}{4\pi\varepsilon_0 r_1^3}$$

$$\mathbf{H} = \frac{i\omega}{4\pi}\,\frac{[\mathbf{r}\times\mathbf{p}]}{r^3} + \frac{i\omega}{4\pi}\,\frac{[\mathbf{r}_1\times\mathbf{p}_1]}{r_1^3}$$

We can find the density of the energy flux by using the solution of Problem 98:

$$\mathbf{S} = \frac{\zeta'}{2}\sqrt{\frac{\mu_0}{\varepsilon_0}}\,|H_t|^2\,\mathbf{n} \tag{1}$$

It is seen from Fig. 58 that

$$\mathbf{r}_1 = 2\mathbf{d} + \mathbf{r}, \quad \mathbf{p}_1 = \frac{(\mathbf{p}\cdot\mathbf{d})\,\mathbf{d}}{d^2} - p_t\mathbf{t}$$

H_t is the projection of the magnetic field on the conducting plane. That is why on the surface of the plane

$$H_t = \frac{2i\omega}{\mu_0 c}\,\frac{p\cos\alpha}{r^3} \tag{2}$$

Substituting formula (2) into (1) and integrating (1) over the surface of the conducting plane, we get the energy absorbed by the plane in one second:

$$\frac{d\mathscr{E}}{dt} = -\int \mathbf{S}\,d\mathbf{s} = \frac{\omega^2 p^2 \cos^2\alpha}{16\pi\varepsilon_0 c d^2}$$

The energy loss of a dipole in a vacuum is less by a factor of about $(\lambda/d)^2$ than that of the dipole on a conducting plane.

122. Let the dipole be in the origin of coordinates. We seek the solution in the following form:

in medium *1*

$$\mathbf{H}_1 = \frac{[\ddot{\mathbf{p}}\times\mathbf{n}]}{4\pi c r} + \frac{[\ddot{\mathbf{p}}_1\times\mathbf{n}_1]}{4\pi c r_1}$$

$$\mathbf{E}_1 = \frac{[[\ddot{\mathbf{p}}\times\mathbf{n}]\times\mathbf{n}]}{4\pi\varepsilon_0\sqrt{\varepsilon_1}\,c^2 r} + \frac{[[\ddot{\mathbf{p}}_1\times\mathbf{n}_1]\times\mathbf{n}_1]}{4\pi\varepsilon_0\sqrt{\varepsilon_1}\,c^2 r_1};$$

in medium *2*

$$\mathbf{H}_2 = \frac{[\ddot{\mathbf{p}}_2\times\mathbf{n}]}{4\pi c r}$$

$$\mathbf{E}_2 = \frac{[[\ddot{\mathbf{p}}_2\times\mathbf{n}]\times\mathbf{n}]}{4\pi\varepsilon_0\sqrt{\varepsilon_2}\,c^2 r}$$

where r is the distance from the dipole to the observation point, $\mathbf{n}$ a unit vector directed towards the point of observation, $\mathbf{r}_1 = \mathbf{r} - 2\mathbf{d}$, $\mathbf{n}_1 = \mathbf{r}_1/r_1$, and $\mathbf{p}_1$ and $\mathbf{p}_2$ can be found from the boundary conditions

$$[\ddot{\mathbf{p}}\times\mathbf{n}]\,\mathbf{t} + [\ddot{\mathbf{p}}_1\times\mathbf{n}_1]\,\mathbf{t} = [\ddot{\mathbf{p}}_2\times\mathbf{n}]\,\mathbf{t}$$

$$\frac{[[\ddot{\mathbf{p}}\times\mathbf{n}]\times\mathbf{n}]\,\mathbf{t}}{\sqrt{\varepsilon_1}} + \frac{[[\ddot{\mathbf{p}}_1\times\mathbf{n}_1]\times\mathbf{n}_1]\,\mathbf{t}}{\sqrt{\varepsilon_1}} = \frac{[[\ddot{\mathbf{p}}_2\times\mathbf{n}]\times\mathbf{n}]\,\mathbf{t}}{\sqrt{\varepsilon_2}}$$

where t is a unit vector in the plane of the interface. Let the latter be the xy-plane. Solving these equations for $\mathbf{p}_1$ and $\mathbf{p}_2$, we get

$$p_{1x}=\frac{\sqrt{\varepsilon_1}-\sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1}+\sqrt{\varepsilon_2}}\,p_x,\quad p_{1y}=\frac{\sqrt{\varepsilon_1}-\sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1}+\sqrt{\varepsilon_2}}\,p_y$$

$$p_{1z}=\frac{\sqrt{\varepsilon_2}-\sqrt{\varepsilon_1}}{\sqrt{\varepsilon_1}+\sqrt{\varepsilon_2}}\,p_z,\quad \mathbf{p}_2=\frac{2\sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1}+\sqrt{\varepsilon_2}}\,\mathbf{p}.$$

123. In determining the long-wave part of the radiation spectrum we can assume the charged particle to bounce off the plane instantly, and so we can use formulas (II-72) and (II-73).

The time dependence of the particle's velocity is given by the following relationships:

$$\mathbf{v}=v_x\mathbf{i}+v_y\mathbf{j}+v_z\mathbf{k}\quad\text{for}\quad t<0$$

$$\mathbf{v}=v_x\mathbf{i}+v_y\mathbf{j}-v_z\mathbf{k}\quad\text{for}\quad t>0$$

Whence

$$\ddot{\mathbf{p}}=e\dot{\mathbf{v}}=2ev_z\delta(t)\,\mathbf{k}$$

where $\mathbf{k}$ is the unit vector normal to the plane. According to formula (II-72), we find that for great distances from the charge

$$\mathbf{E}(\mathbf{r},\,t)=\frac{e\delta(t)}{2\pi\varepsilon_0c^2r}\,v_z\,[[\mathbf{k}\times\mathbf{n}]\times\mathbf{n}]$$

Now, since

$$\delta(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-i\omega t}\,d\omega,$$

we find that

$$\mathbf{E}(\mathbf{r},\,\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathbf{E}(\mathbf{r},\,t)\,e^{i\omega t}\,dt=\frac{ev_z}{2\pi\varepsilon_0c^2r}\,[[\mathbf{k}\times\mathbf{n}]\times\mathbf{n}]$$

The energy radiated per unit of solid angle per unit of frequency interval is

$$\frac{dJ(\omega)}{d\Omega}=\frac{e^2v^2}{4\pi^3c^3\varepsilon_0}\sin^2\theta$$

where θ is the angle between the direction of propagation of waves and $\mathbf{k}$.

The radiation maximum lies in the plane of reflection. There is no radiation in the plane normal to the plane of reflection.

The energy radiated in all directions per unit of frequency interval is

$$J(\omega)=\frac{2e^2v^2}{3\pi^2c^3\varepsilon_0}$$

which shows that at low frequencies (long waves) the radiated energy does not depend on frequency.

125. According to the law of addition of velocities in special relativity,

$$u=\frac{v+\frac{c}{n}}{1+\frac{v}{cn}}$$

when $v \ll c$,

$$u=\frac{c}{n}+v\left(1-\frac{1}{n^2}\right).$$

126. $E=mc^2+\frac{p^2}{2m}-\frac{1}{8}\frac{p^4}{m^3c^2}.$

127. Assume that the electric field is directed along the x-axis, and choose the y-axis in such a way that the particle moves in the xy-plane when $z=0$. The law of motion is then written in the form

$$\frac{dp_x}{dt}=e\,|\mathbf{E}|,\quad \frac{dp_y}{dt}=0$$

After integrating we get

$$p_x=e\,|\mathbf{E}|\,t,\quad p_y=p_0$$

The condition $v_x=p_xc^2/E$, where E is the energy of the particle, yields

$$\frac{dx}{dt}=\frac{e\,|\mathbf{E}|\,tc^2}{[E_0^2+c^2p_0^2+(ce\,|\mathbf{E}|\,t)^2]^{1/2}}=\frac{e\,|\mathbf{E}|\,tc^2}{[E_{01}^2+(ce\,|\mathbf{E}|\,t)^2]^{1/2}}$$

$$\frac{dy}{dt}=\frac{p_0c^2}{[E_{01}^2+(ce\,|\mathbf{E}|\,t)^2]^{1/2}}$$

where $E_0=mc^2$, $E_{01}=c\,(m^2c^2+p_0^2)^{1/2}$. This gives us the path of the particle

$$x=\frac{1}{e\,|\mathbf{E}|}\,[E_{01}^2+(ce\,|\mathbf{E}|\,t)^2]^{1/2}+x_0$$

$$y=\frac{p_0c}{e\,|\mathbf{E}|}\ln\left[t+\sqrt{t^2+\left(\frac{E_{01}}{ce\,|\mathbf{E}|}\right)^2}\,\right]+y_0$$

When $E_0 \gg ce|\mathbf{E}|t$,

$$x=\frac{e|\mathbf{E}|}{2m}t^2+x_0, \quad y=\frac{p_0}{m}t.$$

129. The law of motion of the particle in cylindrical coordinates is

$$\frac{d}{dt}\left(\frac{m\dot{r}}{\sqrt{1-v^2/c^2}}\right)=\frac{mr\dot{\varphi}^2}{\sqrt{1-v^2/c^2}}-\frac{e_1e_2r}{4\pi\varepsilon_0(r^2+z^2)^{3/2}} \quad (1)$$

$$\frac{d}{dt}\left(\frac{mr^2\dot{\varphi}}{\sqrt{1-v^2/c^2}}\right)=0 \quad (2)$$

$$\frac{d}{dt}\left(\frac{m\dot{z}}{\sqrt{1-v^2/c^2}}\right)=-\frac{e_1e_2z}{4\pi\varepsilon_0r^3} \quad (3)$$

$$\frac{d}{dt}\left(\frac{mc^2}{\sqrt{1-v^2/c^2}}\right)=-\frac{e_1e_2\dot{r}}{4\pi\varepsilon_0r^2} \quad (4)$$

We can satisfy Eq. (3) if we put $z = 0$, which corresponds to the motion of the particle in plane $z = 0$. The remaining equations are

$$\frac{d}{dt}\left(\frac{m\dot{r}}{\sqrt{1-v^2/c^2}}\right)=\frac{mr\dot{\varphi}^2}{\sqrt{1-v^2/c^2}}-\frac{e_1e_2}{4\pi\varepsilon_0r^2}$$

$$\frac{d}{dt}\left(\frac{mr^2\dot{\varphi}}{\sqrt{1-v^2/c^2}}\right)=0$$

$$\frac{d}{dt}\left(\frac{mc^2}{\sqrt{1-v^2/c^2}}\right)=\frac{e_1e_2\dot{r}}{4\pi\varepsilon_0r^2}$$

Equation (2) gives one integral of motion—the angular momentum—and, hence, the law of conservation of angular momentum:

$$\frac{mr^2\dot{\varphi}}{\sqrt{1-v^2/c^2}}=L=\text{constant}$$

Equation (4) gives the law of conservation of energy:

$$\frac{mc^2}{\sqrt{1-v^2/c^2}}-\frac{e_1e_2}{4\pi\varepsilon_0r}=E=\text{constant}$$

Eliminating t in the first equation, we get

$$\frac{d^2}{d\varphi^2}\left(\frac{1}{r}\right)+(1-\rho^2)\frac{1}{r}=\frac{e_1e_2E}{4\pi\varepsilon_0L^2c^2}$$

where $\rho=\frac{e_1e_2}{4\pi\varepsilon_0Lc}$.

The solution of this equation is

$$r=\frac{p}{1+\varepsilon\cos\{\sqrt{1-\rho^2}\,(\varphi-\varphi_0)\}}$$

where $p=\frac{L^2c^2-\frac{e_1^2e_2^2}{16\pi^2\varepsilon_0^2}}{4\pi\varepsilon_0e_1e_2E}$, and ε and φ_0 are constants of integration. We can put $\varphi_0=0$ by choosing an appropriate reference point.

The path of the moving particle is not closed. It can be obtained by slowly rotating an ellipse in its plane.

130. $E_1=c^2\frac{M^2+m_1^2-m_2^2}{2M}$, $E_2=c^2\frac{M^2-m_1^2+m_2^2}{2M}$.

131. $T_\mu=4$ MeV, $T_\nu=29.8$ MeV.

132. Let the second frame of reference move along the x-axis of the first. Since p_x, p_y, p_z, $i\frac{E}{c}$ are components of a 4-vector, we can use the law of transformation of 4-vectors (II-79) to find that

$$p_x'=\frac{p_x-\frac{v}{c^2}E}{\sqrt{1-v^2/c^2}},\quad p_y'=p_y,\quad p_z'=p_z$$

$$E'=\frac{E-vp_x}{\sqrt{1-v^2/c^2}}.$$

133. Let the decaying particle move along the x-axis, and the particle with the energy E_1 move at an angle θ_1 to the same axis. From the solution to the previous problem it follows that

$$\cos\theta_1=\frac{p_{1x}}{p_1}=\frac{E_1-E_{01}(1-v^2/c^2)^{1/2}}{v\left(\frac{E_1^2}{c^2}-m_1^2c^2\right)^{1/2}}$$

where E_{01} is the energy of particle *1* in the laboratory frame of reference, and m_1 is the mass of the particle. A similar expression can be found for angle θ_2.

134. $\tan\theta' = \dfrac{u'(1-v^2/c^2)^{1/2}\sin\theta}{v+u'\cos\theta}$, where θ' and θ are angles between the directions of the velocities in the two reference frames and the x- and x'-axes, respectively.

135. $\cos\theta_1 = \dfrac{E_1'(E_1+m_1c^2)-E_1m_2c^2-m_1^2c^4}{c^2p_0p_1}$

$$\cos\theta_2 = \frac{(E_1+m_2c^2)(E_2'-m_2c^2)}{c^2p_0p_2}$$

where $p_0 = \dfrac{1}{c}(E_1^2-m_1^2c^4)^{1/2}$.

136. $\omega = \dfrac{\omega_0}{1+\dfrac{\hbar\omega_0}{mc^2}(1-\cos\theta)}$, where ω_0 is the frequency of the photon before impact, and $\hbar$ is the Planck constant h divided by 2π.

137. $\omega = \dfrac{\Delta E}{\hbar}\left(1-\dfrac{\Delta E}{2mc^2}\right)$.

139. Consider a reference frame that moves together with the mirror. Assume that the x-axis of that system is directed along the velocity of the mirror (opposite the normal). For such a system the laws of reflection in a stationary medium are valid, i.e. the frequency will not change in the course of reflection. The laws of reflection are:

$$k'_x = -k'_{1x}, \quad k'_y = k'_{1y}, \quad k'_z = k'_{1z}, \quad \omega' = \omega'_1$$

where $\mathbf{k}'$ and ω' are the wave vector and frequency of the incident wave, and $\mathbf{k}'_1$ and ω'_1 are the same quantities for the reflected wave. Since the four numbers k_x, k_y, k_z, $i\dfrac{\omega}{c}$ are components of a 4-vector, we can use the law of transformation of 4-vectors to obtain the relationships valid for a reference frame in which the mirror moves:

$$k_x + k_{1x} = \frac{\beta}{c}(\omega+\omega_1)$$

$$k_x - k_{1x} = \frac{1}{v}(\omega-\omega_1)$$

$$k_y = k_{1y}$$

$$k_z = k_{1z}$$

Solving these equations, we find that

$$\omega_1 = \omega \frac{1+\beta^2-2\beta\cos\theta}{1-\beta^2}$$

$$\cos\theta_1 = \frac{2\beta-(1+\beta^2)\cos\theta}{1+\beta^2-2\beta\cos\theta}$$

where θ and θ_1 are the angles between the x-axis and the lines of propagation of the incident and reflected waves, respectively.

141. If we find at least one such frame of reference, any other frame that moves parallel to **E** and **H** will have the same property. For this reason we must find one frame of reference that moves with a velocity which is normal to **E** or **H** and directed, say, along the x-axis. Since in such a frame the fields are still parallel, we have $H'_x = E'_x = 0$ and $E'_y H'_z - E'_z H'_y = 0$. The law of transformation of fields brings us to the final result:

$$\frac{v/c}{1+v^2/c^2} = \frac{[\mathbf{E}\times\mathbf{H}]}{\varepsilon_0 E^2 + \mu_0 H^2}.$$

143. $$T_{xx} = \frac{1}{1-v^2/c^2}\left(T'_{xx} - \frac{2iv}{c}T'_{4x} - \frac{v^2}{c^2}T'_{44}\right)$$

$$T'_{yy} = T_{yy}, \quad T_{yz} = T'_{yz}$$

$$T_{xy} = \frac{1}{(1-v^2/c^2)^{1/2}}\left(T'_{xy} - \frac{iv}{c}T'_{4y}\right)$$

$$T_{4x} = \frac{i}{c(1-v^2/c^2)^{1/2}}\left[-icT'_{4x}\left(1+\frac{v^2}{c^2}\right) - vT'_{44} + vT'_{xx}\right]$$

$$T_{4y} = \frac{1}{(1-v^2/c^2)^{1/2}}\left(T'_{4y} + \frac{iv}{c}T'_{xy}\right)$$

$$T_{44} = \frac{1}{1-v^2/c^2}\left(T'_{44} + 2\frac{iv}{c}T'_{4x} - \frac{v^2}{c^2}T'_{xx}\right).$$

144. $$\mathbf{E}'_{\parallel} = \mathbf{E}_{\parallel}, \quad \mathbf{H}'_{\parallel} = \mathbf{H}_{\parallel}, \quad \mathbf{E}'_{\perp} = \frac{\mathbf{E}_{\perp} + [\mathbf{v}\times\mathbf{B}]}{(1-v^2/c^2)^{1/2}},$$

$$\mathbf{H}'_{\perp} = \frac{\mathbf{H}_{\perp} - \varepsilon_0[\mathbf{v}\times\mathbf{E}]}{(1-v^2/c^2)^{1/2}}.$$

Here the symbols $\parallel$ and $\perp$ stand for 'parallel' and 'normal' to velocity **v**.

SECTION III

1. As the particle moves, its momentum p remains constant and is equal to $-p$ after the particle is reflected from the wall. Thus

$$\oint p\,dx = 2p\int_0^a dx = nh$$

whence

$$p_n = \frac{nh}{2a} \quad \text{and} \quad E_n = \frac{n^2h^2}{8ma^2}.$$

2. Using the fact that the energy of the oscillator is an integral of motion

$$E = \frac{p^2}{2m} + \frac{m\omega^2x^2}{2} = \frac{m\omega^2A^2}{2}$$

where $\omega = (k/m)^{1/2}$ is the frequency of oscillations and A is the amplitude of oscillations, we get

$$p = \sqrt{m^2\omega^2(A^2 - x^2)}$$

Substituting $x = A\cos\varphi$, we find that

$$\oint p\,dx = mA^2\omega\pi = nh$$

Thus the allowed amplitudes and energies are

$$A_n^2 = \frac{nh}{m\pi\omega} \quad \text{and} \quad E_n = \frac{mA_n^2\omega^2}{2} = n\hbar\omega.$$

5. The motion of the electron can be described by two generalized coordinates, r and φ, and two generalized momenta, $p_r = m\dot{r}$ and $p_\varphi = mr^2\dot{\varphi}$. In a central field there are two integrals of motion: the component of angular momentum along the axis of rotation

$$p_\varphi = mr^2\dot{\varphi} = L$$

and the total energy

$$E = \frac{m\dot{r}^2}{2} + \frac{mr^2\dot{\varphi}^2}{2} - \frac{Ze^2}{r}$$

The last relationship yields

$$p_r = \sqrt{2m\left[E + \frac{Ze^2}{r} - \frac{L^2}{2mr^2}\right]}$$

Since $\dot{r} = 0$ at the extremal values of r, it follows that $r_{\max}$ and $r_{\min}$ are the roots of the radicand. The Bohr-Sommerfeld quantization rule gives

$$\oint p_\varphi\, d\varphi = L \times 2\pi = n_\varphi h \quad \text{and} \quad 2\int_{r_{\min}}^{r_{\max}} p_r\, dr = n_r h$$

The second integral is of the form $I = \int_{r_1}^{r_2} \sqrt{A + 2B/r + C/r^2}\; dr$, where $\sqrt{A} = +i\sqrt{-2mE}$ $(E < 0)$ and $\sqrt{C} = -iL = -in_\varphi\hbar$. We can calculate it in the following way. As we have already mentioned, if for r we substitute r_1 or r_2, the radicand turns zero; the radicand is positive if $r_1 < r < r_2$. Consequently, $A < 0$ and $C < 0$, since

$$A + \frac{2B}{r} + \frac{C}{r^2} = \frac{A(r - r_1)(r - r_2)}{r^2}$$

$$= C\left(\frac{1}{r} - \frac{1}{r_1}\right)\left(\frac{1}{r} - \frac{1}{r_2}\right) > 0$$

Denote by $f(z)$ the function $\sqrt{A + \frac{2B}{z} + \frac{C}{z^2}}$, where z is a complex variable. Obviously, $z = r_1$ and $z = r_2$ are the branch points of this function.

To calculate I let us cut the complex plane along the segment $r_1 - r_2$. We choose $f < 0$ just above the cut and $f > 0$ just below. We then evaluate the integral around a contour consisting of the upper and lower sides of the cut (from r_1 to r_2 along the lower side and from r_2 to r_1 along the upper). We denote the contour by l and have

$$\int_l f(z)\, dz = \int_{r_1}^{r_2} |f|\, dr + \int_{r_2}^{r_1} [-|f|]\, dr = 2I$$

Point $z = 0$ is the pole of $f(z)$. Hence, l can be distorted away from the singularities, so that points $z = r_1$ and $z = r_2$ are inside the contour and $z = 0$ is outside. If we

consider a contour L in the neighbourhood of $z=\infty$ and a contour λ around $z=0$, according to Cauchy's theorem,

$$\int_L f(z)\,dz=\int_l f(z)\,dz+\int_\lambda f(z)\,dz$$

with all the contours by-passed in the positive direction (counterclockwise).

To calculate the integrals along L and λ, let us expand $f(z)$ near $z=\infty$ and $z=0$.

For $z=\infty$

$$f(z)=\sqrt{A}\left(1+\frac{2B}{Az}+\frac{C}{Az^2}\right)^{1/2}\cong\sqrt{A}\left(1+\frac{B}{Az}+\dots\right)$$

When we pass from points $z<r_2$ on the lower side of the cut $\left(\text{where } f=\frac{1}{z}\sqrt{A(z-r_1)(z-r_2)}>0 \text{ and } \arg f=0\right)$ to points $z>r_2$, the argument of $z-r_2$ increases by π. Consequently, $\arg f$ changes from 0 to $\pi/2$. Thus in the above expression we must take $\sqrt{A}=+i\,|\sqrt{A}|$, since $\arg\left(1+\frac{B}{Az}+\dots\right)=0$.

For $z=0$

$$f(z)=\frac{\sqrt{C}}{z}\left(1+\frac{2Bz}{C}+\frac{Az^2}{C}\right)^{1/2}\cong\frac{\sqrt{C}}{z}\left(1+\frac{Bz}{C}+\dots\right)$$

Now when we pass from points $z>r_1$ on the lower side of the cut to points $z<r_1$, the argument of $z-r_1$ decreases by π, since we pass in the negative direction (clockwise). Hence, $\sqrt{C}=-i\,|\sqrt{C}|$.

Using the theory of residues, we have

$$\int_L f(z)\,dz\cong\sqrt{A}\int_L\left(1+\frac{B}{Az}+\dots\right)dz=2\pi i\frac{B}{\sqrt{A}}$$

$$\int_\lambda f(z)\,dz\cong\sqrt{C}\int_\lambda\frac{dz}{z}\left(1+\frac{Bz}{C}+\dots\right)=2\pi i\sqrt{C}$$

and finally

$$I=\frac{1}{2}\left[\int_L f(z)\,dz-\int_\lambda f(z)\,dz\right]=\pi i\left(\frac{B}{\sqrt{A}}-\sqrt{C}\right)$$

If we introduce the quantity $n = n_\varphi + n_r$ and substitute the values of $\sqrt{A}$, B, and $\sqrt{C}$, we get

$$2\pi i\left(\frac{me^2}{i\sqrt{2m|E|}} + in_\varphi\hbar\right) = n_r h$$

and

$$E_n = -\frac{me^4}{2\hbar^2 n^2}.$$

6. Let us denote the position and the mass of the nucleus by $\mathbf{r}_1$ and m_1, and those of the electron by $\mathbf{r}_2$ and m_2. We can then separate the variables by introducing $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$ and $\mathbf{R} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2}$.

The centre of mass (with a radius vector $\mathbf{R}$) moves like a free particle of mass $M = m_1 + m_2$ in the given limits, i.e. $0 < x < a$, $0 < y < b$, $0 < z < c$. As for the relative motion of the electron, this reduces to the motion of a particle with mass μ $\left(\text{the reduced mass of the system } \mu = \frac{m_1 m_2}{m_1 + m_2}\right)$ about a fixed centre (see Problem 5). The centre-of-mass momenta, P_x, P_y, P_z, are quantized in the same manner as in Problem 1. Thus

$$E_{n_1 n_2 n_3 n} = \frac{\mathbf{P}^2}{2M} + E_{\text{rel}} = \frac{h^2}{2M}\left(\frac{n_1^2}{a^2} + \frac{n_2^2}{b^2} + \frac{n_3^2}{c^2}\right) - \frac{\mu e^4}{2\hbar^2 n^2}.$$

7. The particle's position z and its momentum p enter the total energy, which is an integral of motion, in the following manner:

$$E = \frac{p^2}{2m} + mgz = mgH$$

which yields

$$p = \pm\sqrt{2m(E - mgz)}$$

Using the quantization rule

$$\oint p\,dz = 2\int_0^H \sqrt{2m(E - mgz)}\,dz = nh$$

enables us to find H_n and E_n:

$$H_n = \left(\frac{3nh}{4m\sqrt{2g}}\right)^{2/3}, \quad E_n = \left(\frac{3\sqrt{m}\,gnh}{4\sqrt{2}}\right)^{2/3}.$$

8. (a) Applying the operator twice to an arbitrary function ψ, we get $\left(\frac{d}{dx}+x\right)^2\psi=\left(\frac{d}{dx}+x\right)\left(\frac{d\psi}{dx}+x\psi\right)=$ $=\frac{d^2\psi}{dx^2}+2x\frac{d\psi}{dx}+x^2\psi+\psi$ and, consequently, $\left(\frac{d}{dx}+x\right)^2=$ $=\frac{d^2}{dx^2}+2x\frac{d}{dx}+x^2+1$;

(b) $\left(\frac{d}{dx}+\frac{1}{x}\right)^3=\frac{d^3}{dx^3}+\frac{3}{x}\frac{d^2}{dx^2}$;

(c) $\left(x\frac{d}{dx}\right)^2=x^2\frac{d^2}{dx^2}+x\frac{d}{dx}$;

(d) $\left(\frac{d}{dx}x\right)^2=x^2\frac{d^2}{dx^2}+3x\frac{d}{dx}+1$;

(e) It is obvious that $[i\hbar\nabla+\mathbf{A}]^2\psi=(i\hbar\nabla+\mathbf{A})\times$ $\times(i\hbar\,\mathrm{grad}\,\psi+\mathbf{A}\psi)=-\hbar^2\Delta\psi+i\hbar\,[(\nabla\cdot\mathbf{A})+(\mathbf{A}\cdot\nabla)]\,\psi+\mathbf{A}^2\psi$. Taking into consideration that $(\nabla\cdot\mathbf{A})\,\psi=(\mathbf{A}\cdot\mathrm{grad}\,\psi)+$ $+\psi\,\mathrm{div}\,\mathbf{A}$, we come to the final result:

$$(i\hbar\nabla+\mathbf{A})^2=-\hbar\Delta+2i\hbar\,(\mathbf{A}\cdot\nabla)+i\hbar\,\mathrm{div}\,\mathbf{A}+\mathbf{A}^2;$$

(f) $(\hat{L}-\hat{M})(\hat{L}+\hat{M})=\hat{L}^2-\hat{M}^2+(\hat{L}\hat{M}-\hat{M}\hat{L})$.

9. (a) Applying the operator $\frac{d}{dx}x-x\frac{d}{dx}$ to an arbitrary function, we get

$$\left(\frac{d}{dx}x-x\frac{d}{dx}\right)\psi=\frac{d}{dx}(x\psi)-x\frac{d\psi}{dx}=\psi$$

or

$$\frac{d}{dx}x-x\frac{d}{dx}=1;$$

(b) $i\hbar\,[(\nabla\cdot\mathbf{A})-(\mathbf{A}\cdot\nabla)]=i\hbar\,\mathrm{div}\,\mathbf{A}$;

(c) $\frac{\partial}{\partial\varphi}f-f\frac{\partial}{\partial\varphi}=\frac{\partial f}{\partial\varphi}$.

10. (a) Let us define the sought operator by the equality

$$\hat{T}_a\psi(x)=\psi(x+a)$$

We then express $\psi(x+a)$ as a power series in a:

$$\psi(x+a)=\psi(x)+a\frac{d\psi}{dx}+\frac{a^2}{2!}\frac{d^2\psi}{dx^2}+\ldots=\sum_{n=0}^{\infty}\frac{a^n}{n!}\frac{d^n}{dx^n}\psi(x)$$

Noting that $\sum\limits_{n=0}^{\infty}\frac{x^n}{n!}=e^x$, we get $\hat{T}_a=e^{a\frac{d}{dx}}$;

(b) We define $\hat{T}_a$ as $\hat{T}_{\mathbf{a}}\psi(\mathbf{r}+\mathbf{a})$ and get $T_{\mathbf{a}}=e^{(\mathbf{a}\cdot\nabla)}$.

$$\hat{T}_\alpha=\sum_{n=0}^{\infty}\left(\alpha\frac{d}{d\varphi}\right)^n\cdot\frac{1}{n!}=e^{\alpha\frac{d}{d\varphi}}.$$

11. (a) According to the definition of a hermitian conjugate operator,

$$\int_{-\infty}^{\infty}\psi_1^*\frac{d\psi_2}{dx}\,dx=\int_{-\infty}^{\infty}\psi_2\left[\left(\frac{d}{dx}\right)^+\psi_1\right]^*dx$$

provided that $\int\limits_{-\infty}^{\infty}|\psi_1|^2\,dx$ and $\int\limits_{-\infty}^{\infty}|\psi_2|^2\,dx$ exist, which implies that ψ_1 and ψ_2 are zero as $x\to\pm\infty$. If in the right-hand side of the equality we integrate by parts:

$$\int_{-\infty}^{\infty}\psi_1^*\frac{d\psi_2}{dx}\,dx=\psi_1^*\psi_2\Big|_{-\infty}^{\infty}-\int_{-\infty}^{\infty}\psi_2\frac{d\psi_1^*}{dx}\,dx=\int_{-\infty}^{\infty}\psi_2\left(-\frac{d\psi_1}{dx}\right)^*dx$$

we find that $\left(\frac{d}{dx}\right)^+=-\frac{d}{dx}$.

(b) In a similar manner we get

$$\left(\frac{\partial^n}{\partial x^n}\right)^+=(-1)^n\frac{\partial^n}{\partial x^n}.$$

13. The operator $\hat{T}_{\mathbf{a}}^+$ is defined by the relation

$$\int\psi_1^*(\mathbf{r})\,\hat{T}_{\mathbf{a}}\psi_2(\mathbf{r})\,d\tau=\int\psi_2(\mathbf{r})\,[\hat{T}_{\mathbf{a}}^+\psi_1(\mathbf{r})]^*\,d\tau$$

In order to find this operator, we introduce a change in variables, $\mathbf{r}+\mathbf{a}=\mathbf{r}'$, in the expression defining $\hat{T}_{\mathbf{a}}$:

$$I=\int \psi_1^*(\mathbf{r})\hat{T}_{\mathbf{a}}\psi_2(\mathbf{r})\,d\tau=\int \psi_1^*(\mathbf{r})\,\psi_2(\mathbf{r}+\mathbf{a})\,d\tau$$

$$=\int \psi_1^*(\mathbf{r}'-\mathbf{a})\,\psi_2(\mathbf{r}')\,d\tau'$$

The last equality stems from the fact that the integration is carried over the entire space and thus the value of the integral remains the same.

But since $\psi_1(\mathbf{r}-\mathbf{a})=\hat{T}_{-\mathbf{a}}\psi_1(\mathbf{r})$, we can write

$$I=\int \psi_2(\mathbf{r})\,[\hat{T}_{-\mathbf{a}}\psi_1(\mathbf{r})]^*\,d\tau$$

i.e. $\hat{T}_{\mathbf{a}}^+=\hat{T}_{-\mathbf{a}}$.

14. By definition, $e^{i\alpha\frac{\partial}{\partial\varphi}}=\sum\limits_{n=0}^{\infty}\frac{\left(i\alpha\frac{\partial}{\partial\varphi}\right)}{n!}$. According to Problem 11(b), the hermitian conjugate to $\frac{\partial^n}{\partial\varphi^n}$ is $(-1)^n\frac{\partial^n}{\partial\varphi^n}$. Consequently,

$$\left[\left(i\frac{\partial}{\partial\varphi}\right)^n\right]^+=(-i)^n\left(-\frac{\partial}{\partial\varphi}\right)^n=\left(i\frac{\partial}{\partial\varphi}\right)^n$$

is a hermitian operator. Thus the hermiticity of the original operator follows:

$$\left(e^{i\alpha\frac{\partial}{\partial\varphi}}\right)^+=e^{i\alpha\frac{\partial}{\partial\varphi}}\quad(\text{if } \alpha=\alpha^*).$$

15. By definition,

$$\int \psi_1^*\hat{A}\hat{B}\psi_2\,d\tau=\int \psi_2\,[(\hat{A}\hat{B})^+\psi_1]^*\,d\tau$$

Let us consider the first integral. We denote $\hat{B}\psi_2=\psi_3$ and introduce $\hat{A}^+$. We can then write

$$\int \psi_1^*\hat{A}\psi_3\,d\tau=\int \psi_3\,(\hat{A}^+\psi_1)^*\,d\tau$$

Introduce a new function ψ_4 such that $\psi_4=\hat{A}^+\psi_1$ and then return to ψ_2. We can rewrite the last equality in a form

convenient for a transformation via $\hat{B}^+$:

$$\int \psi_3 (\hat{A}^+\psi_1)^* \, d\tau = \int \psi_4^* \hat{B} \psi_2 \, d\tau = \int \psi_2 (\hat{B}^+\psi_4)^* \, d\tau$$
$$= \int \psi_2 (\hat{B}^+\hat{A}^+\psi_1)^* \, d\tau$$

Thus

$$(\hat{A}\hat{B})^+ = \hat{B}^+\hat{A}^+.$$

16. *Hint.* Use the results obtained in Problem 15.

17. *Hint.* Use definitions (III-14) and (III-16).

18. Adding $\pm\hat{M}\hat{L}\hat{M}$ to $\hat{L}\hat{M}^2 - \hat{M}^2\hat{L}$ and taking the common factor $\hat{M}$ out of the differences leftwards and rightwards, we obtain

$$\hat{L}\hat{M}^2 - \hat{M}^2\hat{L} + \hat{M}\hat{L}\hat{M} - \hat{M}\hat{L}\hat{M} = (\hat{L}\hat{M} - \hat{M}\hat{L})\,\hat{M}$$
$$+ \hat{M}\,(\hat{L}\hat{M} - \hat{M}\hat{L}) = 2\hat{M}$$

since $\hat{L}\hat{M} - \hat{M}\hat{L} = 1$.

19. Using the results obtained in Problem 18, we prove that being valid for $n = 2$ the relationship is valid for $n + 1$ as well. Suppose

$$\hat{M}\hat{L}^n - \hat{L}^n\hat{M} = -n\hat{L}^{n-1}$$

Now we form the expression

$$\hat{M}\hat{L}^{n+1} - \hat{L}^{n+1}\hat{M} = \hat{M}\hat{L}^{n+1} - \hat{L}\,(\hat{M}\hat{L}^n + n\hat{L}^{n-1})$$
$$= (\hat{M}\hat{L} - \hat{L}\hat{M})\,\hat{L}^n - n\hat{L}^n = -(n+1)\,\hat{L}^n$$

where we have used the previous equation. Thus the relationship is proved for any n. By definition,

$$f(\hat{L}) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}\,\hat{L}^n$$

Consequently,

$$f(\hat{L})\,\hat{M} - \hat{M}f(\hat{L}) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}\,(\hat{L}^n\hat{M} - \hat{M}\hat{L}^n)$$
$$= \sum_{n=1}^{\infty} \frac{f^{(n)}(0)}{(n-1)!}\,\hat{L}^{n-1}$$

But if we put $n-1=n_1$, we get

$$\sum_{n_1=0}^{\infty}\frac{f^{(n_1+1)}(0)}{n_1!}\hat{L}^{n_1}=\sum_{n_1=0}^{\infty}\frac{[f'(0)]^{(n_1)}}{n_1!}\hat{L}^{n_1}=f'(\hat{L})$$

and in this way

$$f(\hat{L})\hat{M}-\hat{M}f(\hat{L})=f'(\hat{L}).$$

20. In order to prove assertions (a) and (b), we must present $\hat{B}^2$ in the form $\hat{B}\hat{A}\hat{A}^{-1}\hat{B}$. As for (c), let us express $f(\hat{B})$ as a power series in $\hat{B}$, $f(\hat{B})=\sum\limits_{n}^{\infty}\frac{1}{n!}f^{(n)}(0)\hat{B}^n$, and then use (b).

21. First expand $e^{-\zeta\hat{A}}$ in a series and then, in a way similar to that of Problem 19, find the commutator

$$\hat{B}e^{-\zeta\hat{A}}-e^{-\zeta\hat{A}}\hat{B}=\sum_{n=0}^{\infty}\frac{(-\zeta)^n}{n!}(\hat{B}\hat{A}^n-\hat{A}^n\hat{B})=C\zeta e^{-\zeta\hat{A}}$$

And so

$$e^{\zeta\hat{A}}\hat{B}e^{-\zeta\hat{A}}=e^{\zeta\hat{A}}(e^{-\zeta\hat{A}}\hat{B}+C\zeta e^{-\zeta\hat{A}})=\hat{B}+C\zeta.$$

22. If we introduce the operators $\hat{A}=e^{i\zeta\hat{p}/\hbar}$ and $\hat{A}^{-1}=$ $=e^{-i\zeta\hat{p}/\hbar}$ and then use the results obtained in Problem 20(c), we can write

$$e^{i\zeta\hat{p}/\hbar}F(\hat{q})e^{-i\zeta\hat{p}/\hbar}=F(e^{i\zeta\hat{p}/\hbar}\hat{q}e^{-i\zeta\hat{p}/\hbar}) \qquad (1)$$

But since $\hat{p}\hat{q}-\hat{q}\hat{p}=-i\hbar$, we have

$$\hat{q}e^{-i\zeta\hat{p}/\hbar}=e^{-i\zeta\hat{p}/\hbar}(\hat{q}+\zeta) \qquad (2)$$

(by analogy with Problem 19).

Substituting (2) into (1), we get the required relationship.

23. We construct the eigenvalue equations for our operators. For $\frac{d}{dx}$

$$\frac{d\psi}{dx}=\lambda\psi \text{ or } \psi=e^{\lambda x}$$

The finiteness of $\psi(x)$ as $x \to \pm\infty$ implies that $\lambda = i\beta$, where β is a real number. In a similar manner, for $i\frac{d}{dx}$,

$$\psi = e^{-i\lambda x}$$

where λ is a real number (a continuous spectrum).

24. Separating variables in the standard equation $\left(x + \frac{d}{dx}\right)\psi = \lambda\psi$, we arrive at the equation $\frac{d\psi}{\psi} = (\lambda - x)\,dx$. After solving this equation, we get

$$\psi = ce^{\lambda x - x^2/2}$$

Such a solution satisfies the requirements of being finite, continuous, and single-valued for any λ, both real and complex (a continuous spectrum).

25. We seek the solution of $\frac{d\psi}{d\varphi} = \lambda\psi$ in the form

$$\psi = ce^{\lambda\varphi}$$

Because our eigenfunction must be single-valued, the solution must satisfy the condition

$$\psi(\varphi) = \psi(\varphi + 2\pi)$$

Substituting the original solution, we find λ from the condition $e^{\lambda 2\pi} = 1$. Thus $\lambda = im$, where $m = 0, \pm 1, \pm 2, \ldots$.

26. In order to find the solution of

$$\sin\frac{d}{d\varphi}\,\psi = \lambda\psi \tag{1}$$

we express the operator $\sin\frac{d}{d\varphi}$ in a power series:

$$\sin\frac{d}{d\varphi} = \frac{d}{d\varphi} - \frac{1}{3!}\frac{d^3}{d\varphi^3} + \frac{1}{5!}\frac{d^5}{d\varphi^5} - \ldots$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}\,\frac{d^{2k+1}}{d\varphi^{2k+1}} \tag{2}$$

We seek the solution of (1), having in mind (2), in the form $\psi = e^{\alpha\varphi}$, where α can be found from the single-valuedness of the function, $\alpha = im$ ($m = 0, \pm 1, \pm 2, \ldots$; see Problem 25). After substituting this solution into Eq. (1), we

get the eigenvalues

$$\lambda = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} (im)^k = \sin (im).$$

27. Similarly to Problem 26, $\psi = e^{im\varphi}$, $\lambda = \cos m$.
28. Similarly to Problem 26, $\psi = e^{im\varphi}$, $\lambda = a^{-am}$.
29. Introducing a new function $U = x\psi$, we get

$$\frac{d^2U}{dx^2} = \lambda U$$

whose solution is

$$U_1 = C_1 e^{\sqrt{\lambda} x} \quad \text{and} \quad U_2 = C_2 e^{-\sqrt{\lambda} x}$$

For $\lambda = -\beta^2 < 0$ both functions will be finite as $x \to \pm\infty$. But in order to be finite at $x = 0$, $\psi(x)$ must be a linear combination of U_1 and U_2 divided by x,

$$\psi(x) = \frac{C_1 e^{i\beta x} + C_2 e^{-i\beta x}}{x}$$

and the numerator must turn zero together with the denominator (at $x = 0$). Thus

$$C_1 + C_2 = 0 \quad \text{and} \quad \psi(x) = C\frac{\sin \beta x}{x}$$

where β is a real number.

30. We consider the left member in the Poisson bracket as being a product of two operators. Then we do the same with the right member. In each case we use the relation given in the problem and get

$$\{\hat{f}_1 \hat{f}_2, \hat{g}\} = \{\hat{f}_1, \hat{g}\} \hat{f}_2 + \hat{f}_1 \{\hat{f}_2, \hat{g}\} \tag{1}$$

$$\{\hat{f}, \hat{g}_1 \hat{g}_2\} = \{\hat{f}, \hat{g}_1\} \hat{g}_2 + \hat{g}_1 \{\hat{f}, \hat{g}_2\} \tag{2}$$

Now in equations (1) and (2) we set $\hat{f} = \hat{f}_1 \hat{f}_2$ and $\hat{g} = \hat{g}_1 \hat{g}_2$. Using the same relation and keeping to a definite order in the multiplication of operators, we get the same left-hand sides, and the right-hand sides can be expressed in two forms:

from Eq. (1),

$$\{\hat{f}_1 \hat{f}_2, \hat{g}_1 \hat{g}_2\} = \hat{g}_1 \{\hat{f}_1, \hat{g}_2\} \hat{f}_2 + \{\hat{f}_1, \hat{g}_1\} \hat{g}_2 \hat{f}_2 + \hat{f}_1 \hat{g}_1 \{\hat{f}_2, \hat{g}_2\} + \hat{f}_1 \{\hat{f}_2, \hat{g}_1\} \hat{g}_2;$$

from Eq. (2)

$$\{\hat{f}_1\hat{f}_2, \hat{g}_1\hat{g}_2\} = \hat{f}_1\{\hat{f}_2, \hat{g}_1\}\hat{g}_2 + \{\hat{f}_1, \hat{g}_1\}\hat{f}_2\hat{g}_2 + \hat{g}_1\hat{f}_1\{\hat{f}_2, \hat{g}_2\} + \hat{g}_1\{\hat{f}_1, \hat{g}_2\}\hat{f}_2$$

Subtracting one expression from the other, we get a relationship that holds true for any set of four operators $\hat{f}_1$, $\hat{f}_2$, $\hat{g}_1$, $\hat{g}_2$:

$$\{\hat{f}_1, \hat{g}_1\}[\hat{f}_2\hat{g}_2 - \hat{g}_2\hat{f}_2] = [\hat{f}_1\hat{g}_1 - \hat{g}_1\hat{f}_1]\{\hat{f}_2, \hat{g}_2\}$$

The arbitrariness of the operators implies that for any two operators $\hat{f}$ and $\hat{g}$,

$$\{\hat{f}, \hat{g}\} = C\{\hat{f}\hat{g} - \hat{g}\hat{f}\}$$

If we require that $\{\hat{f}, \hat{g}\}^+ = \{\hat{f}, \hat{g}\}$ provided $\hat{f}^+ = \hat{f}$ and $\hat{g}^+ = \hat{g}$ (the hermiticity of the quantum Poisson bracket), we find that $C^* = -C$, or $C = i/\hbar$. The dimension of C must be the one of the reciprocal value of action by analogy with the dimension of the classical Poisson bracket $\{f, g\} = \sum_{i=1}^{n}\left(\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i}\right)$. Thus, $\hbar$ is an arbitrary quantity having the dimension of action, and the quantum Poisson bracket

$$\{\hat{f}, \hat{g}\} = \frac{i}{\hbar}[\hat{f}\hat{g} - \hat{g}\hat{f}].$$

31. $\hat{a}\hat{a}^* - \hat{a}^*\hat{a} = 1.$

32. $\hat{H} = \frac{\hbar\omega}{2}(\hat{a}^*\hat{a} + \hat{a}\hat{a}^*).$

33. We take advantage of the fact that

$$\hat{x}\hat{y} = \hat{y}\hat{x}, \quad \hat{p}_x\hat{p}_y = \hat{p}_y\hat{p}_x, \quad \hat{p}_x\hat{y} - \hat{y}\hat{p}_x = -i\hbar\delta_{xy}$$

and prove the sought commutation relation for the x-component, i.e.

$$\hat{L}_y\hat{L}_z - \hat{L}_z\hat{L}_y = (\hat{z}\hat{p}_x - \hat{x}\hat{p}_z)(\hat{x}\hat{p}_y - \hat{y}\hat{p}_x) - (\hat{x}\hat{p}_y - \hat{y}\hat{p}_x)(\hat{z}\hat{p}_x - \hat{x}\hat{p}_z)$$
$$= (\hat{z}\hat{p}_y - \hat{y}\hat{p}_z)(\hat{p}_x\hat{x} - \hat{x}\hat{p}_x) = i\hbar(\hat{y}\hat{p}_z - \hat{z}\hat{p}_y) = i\hbar\hat{L}_x$$

A cyclic permutation of the indices x, y, z enables us to obtain the other two components, and in vector form

$$[\hat{\mathbf{L}} \times \hat{\mathbf{L}}] = i\hbar\hat{\mathbf{L}}.$$

34. To find the commutation relation for $\hat{\mathbf{L}}^2$ and $\hat{L}_x$ we write

$$\hat{\mathbf{L}}^2\hat{L}_x - \hat{L}_x\hat{\mathbf{L}}^2 = \hat{L}_y^2\hat{L}_x - \hat{L}_x\hat{L}_y^2 + \hat{L}_z^2\hat{L}_x - \hat{L}_x\hat{L}_z^2$$

If we add and subtract $\hat{L}_y\hat{L}_x\hat{L}_y$ and $\hat{L}_z\hat{L}_x\hat{L}_z$, factor out common multipliers leftwards and rightwards, and use the results obtained in Problem 33, we get the needed result.

37. We seek the solution of the equation $i\hbar\frac{\partial\Psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2}$ in the form $\Psi(x, t) = U(x)\varphi(t)$ (this is a particular solution since the variables may be separated). Then

$$\frac{i\hbar}{\varphi}\frac{d\varphi}{dt} = -\frac{\hbar^2}{2mU}\frac{d^2U}{dx^2} = a$$

If we want $U(x)$ to be finite as $|x| \to \infty$, we must put $a > 0$. Denoting $k^2 = \frac{2ma}{\hbar^2}$, we get

$$\varphi(t) = e^{-\frac{i\hbar k^2}{2m}t}, \quad U(x) = e^{ikx}$$

where k is a real number. The general solution is

$$\Psi(x, t) = \int_{-\infty}^{\infty} C(k)\, e^{ikx - i\hbar k^2 t/2m}\, dk.$$

38. The normalization constant A is determined from the condition that $\int_{-\infty}^{\infty} |\Psi(x, 0)|^2\, dx = 1$. Substituting the given wave function, we arrive at an expression for A:

$$|A|^2 \int_{-\infty}^{\infty} e^{-x^2/a^2}\, dx = |A|^2 a\sqrt{\pi} = 1$$

whence

$$|A|^2 = \frac{1}{a\sqrt{\pi}}$$

In order to determine the region where the particle is localized we must find the probability density ρ:

$$\rho = |\Psi(x, 0)|^2 = |A|^2 e^{-x^2/a^2}$$

The peak of this function lies at $x = 0$ and the function rapidly decreases for $|x| > a$. The width of the wave packet represented by this function is of the order of a. The probability current density is

$$j_x = \frac{i\hbar}{2m}\left(\Psi \frac{\partial \Psi^*}{\partial x} - \Psi^* \frac{\partial \Psi}{\partial x}\right) = |A|^2 \frac{\hbar k_0}{m} e^{-\frac{x^2}{a^2}} = \frac{\hbar k_0}{m} \rho$$

The final expression for j_x coincides with the classical formula. The factor ρ is determined only by the real part of the exponent in the wave function, and the quantity $\frac{\hbar k_0}{m}$ (the analog of velocity in classical physics) by its imaginary part.

39. Let us express $\Psi(x, 0)$ in the form of a wave packet (see the solution to Problem 37): $\Psi(x, 0) = \int\limits_{-\infty}^{\infty} C(k)\, e^{ikx} dk$.

Whence

$$C(k) = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} \Psi(x, 0)\, e^{-ikx}\, dx = \frac{A}{2\pi} \int\limits_{-\infty}^{\infty} e^{i(k_0 - k)x - \frac{x^2}{2a^2}}\, dx$$

We then complete squares in the exponent under the integral sign and get

$$C(k) = \frac{A}{2\pi} e^{-\frac{a^2 (k_0 - k)^2}{2}} \int\limits_{-\infty}^{\infty} e^{-\frac{1}{2a^2}[x - i(k_0 - k)a^2]^2}\, dx$$

$$= \frac{Aa}{\sqrt{2\pi}} e^{-\frac{a^2 (k_0 - k)^2}{2}}$$

The function $C(k)$ is nonzero near $k = k_0$, which implies that

$$|C(k)|^2\, dk = \frac{A^2 a^2}{2\pi} e^{-a^2 (k_0 - k)^2}\, dk$$

is proportional to the probability of finding the particle's momentum in the interval $(k, k + dk)$. The width of the wave packet in k-space is $\Delta k \approx \frac{1}{a}$.

40. If we take the wave function from Problem 37, $\Psi(x, t) = \int\limits_{-\infty}^{\infty} C(k)\, e^{ikx - i\hbar k^2 t/2m}\, dk$ and substitute $C(k)$ from Problem 39 into it, we obtain

$$\Psi(x, t) = \frac{Aa}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} \exp\left(ikx - i\,\frac{\hbar k^2 t}{2m} - \frac{a^2 k^2}{2} + a^2 k k_0 - \frac{a^2 k_0^2}{2}\right) dk$$

By completing squares in the exponent and making use of

$$\int\limits_{-\infty}^{\infty} e^{-\alpha k^2}\, dk = \sqrt{\frac{n}{\alpha}}$$

we obtain

$$\Psi(x, t) = \frac{Aa}{\sqrt{2\pi}} \exp\left[\frac{\left(k_0 + \frac{ix}{a^2}\right)^2}{1 + \frac{i\hbar t}{ma^2}}\,\frac{a^2}{2} - \frac{a^2}{2} k_0^2\right]$$

$$\times \int\limits_{-\infty}^{\infty} \exp\left[-\frac{a^2}{2}\left(k\sqrt{1 + \frac{i\hbar t}{ma^2}} - \frac{k_0 + \frac{ix}{a^2}}{\sqrt{1 + \frac{i\hbar t}{ma^2}}}\right)^2\right] dk$$

$$= \frac{A}{\sqrt{1 + \frac{i\hbar t}{ma^2}}} \exp\left[-\frac{x^2 - 2ia^2 x k_0 + \frac{i\hbar a^2 k_0^2}{m}\, t}{2\left(1 + \frac{i\hbar t}{ma^2}\right) a^2}\right]$$

The probability density is, by definition, equal to

$$\rho = |\Psi|^2 = \frac{|A|^2}{\sqrt{1 + \frac{\hbar^2 t^2}{m^2 a^4}}} \exp\left[-\frac{\left(x - \frac{k_0 \hbar t}{m}\right)^2}{\left(1 + \frac{\hbar^2 t^2}{m^2 a^4}\right) a^2}\right]$$

which implies that we are dealing with a wave packet whose peak is traveling with velocity $\frac{\hbar k_0}{m}$ and whose width increases in time as

$$a' = a\sqrt{1+\frac{\hbar^2 t^2}{m^2 a^4}}$$

The probability current density can be represented in the form

$$j_x = \frac{i\hbar}{2m}\left(\Psi\frac{\partial\Psi^*}{\partial x} - \Psi^*\frac{\partial\Psi}{\partial x}\right)$$

$$= \frac{i\hbar}{2m}\left(\Psi^*\Psi\frac{-ik_0-\frac{x}{a^2}}{1-\frac{i\hbar t}{ma^2}} - \Psi^*\Psi\frac{ik_0-\frac{x}{a^2}}{1+\frac{i\hbar t}{ma^2}}\right)$$

$$= \rho\frac{\hbar k_0}{m}\,\frac{1+\frac{\hbar t x}{ma^4 k_0}}{1+\frac{\hbar^2 t^2}{m^2 a^4}}\,.$$

41. $\langle x\rangle = \int\limits_{-\infty}^{\infty} x\,|\psi|^2\,dx = |\mathrm{A}|^2\int\limits_{-\infty}^{\infty} xe^{-x^2/a^2}\,dx = 0$

$$\langle p\rangle = \int\limits_{-\infty}^{\infty}\psi^*\left(-i\hbar\frac{\partial\psi}{\partial x}\right)dx = \int\limits_{-\infty}^{\infty}\psi^*\left(\hbar k_0+\frac{i\hbar x}{a^2}\right)\psi\,dx = \hbar k_0.$$

42. Since $\langle x\rangle = 0$ (see Problem 41),

$$\Delta x = x - \langle x\rangle = x$$

whence

$$\langle\Delta x^2\rangle = \int\limits_{-\infty}^{\infty} x^2|\psi|^2\,dx = |A|^2\int\limits_{-\infty}^{\infty} x^2 e^{-x^2/a^2}\,dx = \frac{|A|^2}{2}\sqrt{\pi}\,a^3$$

It follows from the normalization condition that $|A|^2 = (\pi a)^{-1/2}$ and

$$\langle\Delta x^2\rangle = \frac{a^2}{2}$$

To determine $\langle\Delta p^2\rangle = \hbar^2\langle(k-k_0)^2\rangle$ we can use $|C(k)|^2dk$ from the answer to Problem 39, first normalizing it, i.e. we

proceed as follows:

$$dW(k) = Be^{-a^2(k-k_0)^2}\,dk$$

$$1 = \int_{-\infty}^{\infty} Be^{-a^2(k-k_0)^2}\,dk = B\sqrt{\frac{\pi}{a^2}}$$

$$B = \frac{a}{\sqrt{\pi}}$$

Thus

$$\langle \Delta p^2 \rangle = B\hbar^2 \int_{-\infty}^{\infty} (k-k_0)^2\, e^{-a^2(k-k_0)^2}\,dk$$

$$= B\hbar^2 \left[-\frac{\partial}{\partial (a^2)} \int_{-\infty}^{\infty} e^{-a^2(k-k_0)^2}\,dk \right]$$

$$= B\hbar^2 \frac{\sqrt{\pi}}{2a^3} = \frac{\hbar^2}{2a^2}$$

and we immediately arrive at the uncertainty relation

$$\langle \Delta x^2 \rangle \langle \Delta p^2 \rangle = \frac{a^2}{2}\,\frac{\hbar^2}{2a^2} = \frac{\hbar^2}{4}.$$

43. Let us consider the first ($x < 0$) and third ($x > a$) regions, i.e. the regions where $V = \infty$. Taking the Schrödinger equation in these regions in the form

$$\frac{1}{\psi_{\mathrm{I}}}\,\frac{d^2\psi_{\mathrm{I}}}{dx^2} = \frac{2m}{\hbar^2}(V-E)$$

we see that the function ψ_{I} must turn zero when $V = \infty$ (this can be proved more rigorously if we use the solution to Problem 45).

In the second region ($0 \leqslant x \leqslant a$) the Schrödinger equation comes down to

$$\frac{d^2\psi_{\mathrm{II}}}{dx^2} + \frac{2mE}{\hbar^2}\psi_{\mathrm{II}} = 0$$

If we denote $k^2 = 2mE/\hbar^2$, we can write its solution as $\psi_{\mathrm{II}} = A \sin(kx + \alpha)$. Since the wave function must be continuous at all points in space, for one, in the transition from the first region to the second, $x = 0$, and from the

second to the third, $x = a$, we must assume that $\psi_{\mathrm{I}}(0) = \psi_{\mathrm{II}}(0)$ and $\psi_{\mathrm{II}}(a) = \psi_{\mathrm{III}}(a)$, i.e.

$$A \sin \alpha = 0 \text{ and } A \sin (ka + \alpha) = 0$$

Hence, α is zero and k can take on none but discrete integral values: $k_n = \frac{n\pi}{a}$, where $n = 1, 2, \ldots$. The energy levels are

$$E_n = \frac{n^2\pi^2\hbar^2}{2m}$$

The normalization condition

$$\int_{-\infty}^{\infty} |\psi|^2 dx = \int_0^a |\psi_{\mathrm{II}}|^2 dx = |A|^2 \int_0^a \sin^2 \frac{n\pi x}{a} dx = 1$$

yields $A = \sqrt{2/a}$. The wave function is specified at any point in space:

$$\psi_{\mathrm{I}} = 0, \quad \psi_{\mathrm{II}} = \sqrt{\frac{2}{a}} \sin \frac{n\pi x}{a}, \quad \psi_{\mathrm{III}} = 0.$$

44. In the regions where $V = \infty$ the wave function ψ is zero (see the solution to Problem 43). But in the regions where V is zero the wave function satisfies the equation

$$\Delta\psi + \frac{2mE}{\hbar^2}\psi = 0$$

in which the variables can be separated: $\psi(x, y, z) = \psi_1(x)\psi_2(y)\psi_3(z)$. For all three functions we obtain similar equations as in Problem 43 with analogous boundary conditions:

$$\psi_1(0) = \psi_1(a) = 0$$
$$\psi_2(0) = \psi_2(b) = 0$$
$$\psi_3(0) = \psi_3(c) = 0$$

When we satisfy these conditions and also the normalization condition in the regions where V is zero, the wave function takes the form

$$\psi_{n_1n_2n_3}(x, y, z) = \sqrt{\frac{8}{abc}} \sin \frac{n_1\pi x}{a} \sin \frac{n_2\pi y}{b} \sin \frac{n_3\pi z}{c};$$

in the rest of the space $\psi = 0$.

The energy corresponding to the wave function is

$$E_{n_1n_2n_3}=\frac{\hbar^2\pi^2}{2m}\left(\frac{n_1^2}{a^2}+\frac{n_2^2}{b^2}+\frac{n_3^2}{c^2}\right).$$

45. Let us restrict ourselves to the case when $E<0$. In the first and third regions, after we denote $\varkappa^2 = -2mE/\hbar^2>0$, the Schrödinger equation takes the form

$$\frac{d^2\psi_{\mathrm{I}}}{dx^2}-\varkappa^2\psi_{\mathrm{I}}=0$$

Consequently,

$$\psi_{\mathrm{I}}=Ae^{\varkappa x}+Be^{-\varkappa x},\quad \psi_{\mathrm{III}}=Ce^{-\varkappa x}+De^{\varkappa x}$$

From the requirement that ψ_{I} and ψ_{III} be finite as $x\to\pm\infty$ it follows that $D=B=0$. Then

$$\psi_{\mathrm{I}}=Ae^{\varkappa x},\quad \psi_{\mathrm{III}}=Ce^{-\varkappa x}$$

In the second region we have the equation

$$\frac{d^2\psi_{\mathrm{II}}}{dx^2}+k^2\psi_{\mathrm{II}}=0$$

where $k^2 = 2m(V_0+E)/\hbar^2$. Then

$$\psi_{\mathrm{II}}=\alpha\sin kx+\beta\cos kx$$

At points $x=-a$ and $x=a$ we require that the function and its derivative be continuous. (This is possible since the discontinuity of the potential energy at these points is finite.) Thus,

$$\psi_{\mathrm{I}}(-a)=\psi_{\mathrm{II}}(-a),\qquad \psi_{\mathrm{II}}(a)=\psi_{\mathrm{III}}(a)$$

$$\left.\frac{d\psi_{\mathrm{I}}}{dx}\right|_{x=-a}=\left.\frac{d\psi_{\mathrm{II}}}{dx}\right|_{x=-a},\qquad \left.\frac{d\psi_{\mathrm{II}}}{dx}\right|_{x=a}=\left.\frac{d\psi_{\mathrm{III}}}{dx}\right|_{x=a}$$

For coefficients A, C, α, β we get four homogeneous linear equations:

$$\left.\begin{aligned}Ae^{-\varkappa a}+\alpha\sin ka-\beta\cos ka&=0\\ A\varkappa e^{-\varkappa a}-\alpha k\cos ka-\beta k\sin ka&=0\\ Ce^{-\varkappa a}-\alpha\sin ka-\beta\cos ka&=0\\ C\varkappa e^{-\varkappa a}+\alpha k\cos ka-\beta k\sin ka&=0\end{aligned}\right\}\tag{1}$$

This system has a solution if its determinant is zero, that is, if

$$\varkappa^2 - k^2 + 2k\varkappa \cot 2ka = 0 \qquad (2)$$

Solving equation (2) for $\varkappa$, we get two roots:
(a) $\varkappa = k \tan ka$. Substituting this into (1), we get

$$\alpha = 0, \quad C = A, \quad \beta = \frac{A}{\cos k\alpha} e^{-\varkappa a}$$

and

$$\psi_{\mathrm{I}} = Ae^{\varkappa x}, \quad \psi_{\mathrm{II}} = Ae^{-\varkappa a} \frac{\cos kx}{\cos ka}, \quad \psi_{\mathrm{III}} = Ae^{-\varkappa x}$$

which implies that ψ is an even function: $\psi(x) = \psi(-x)$.
(b) $\varkappa = -k \cot ka$. System (1) yields

$$\alpha = -A \frac{e^{-\varkappa a}}{\sin ka}, \quad \beta = 0, \quad C = -A$$

and the solution

$$\psi_{\mathrm{I}} = Ae^{\varkappa x}, \quad \psi_{\mathrm{II}} = -Ae^{-\varkappa a} \frac{\sin kx}{\sin ka}, \quad \psi_{\mathrm{III}} = -Ae^{-\varkappa x}$$

is an odd function: $\psi(-x) = -\psi(x)$.

Coefficient A is determined by the normalization condition. For example, the first case gives

$$\int_{-\infty}^{\infty} |\psi|^2 \, dx$$

$$= |A|^2 \left[\int_{-\infty}^{-a} e^{2\varkappa x} \, dx + \frac{e^{-2\varkappa a}}{\cos^2 ka} \int_{-a}^{a} \cos^2 kx \, dx + \int_{a}^{\infty} e^{-2\varkappa x} \, dx \right]$$

$$= |A|^2 e^{-2\varkappa a} \left[\frac{1}{\varkappa} + \frac{a}{\cos^2 ka} + \frac{\sin 2ka}{2k \cos^2 ka} \right] = 1$$

If we substitute $k \tan ka$ for $\varkappa$, we find the final expression for A,

$$|A|^{-2} = ae^{-2\varkappa a} \left(1 + \frac{1}{\varkappa a} + \frac{\varkappa^2}{k^2} + \frac{\varkappa}{k^2 a} \right)$$

which holds for the second case as well.

To determine the energy levels we make use of the fact that

$$\varkappa^2 + k^2 = \frac{2mV_0}{\hbar^2} = \frac{C_1^2}{a^2}$$

where C_1 is a dimensionless constant that characterizes the depth of the potential well. Whence,

$$\varkappa a = \sqrt{C_1^2 - (ka)^2}$$

and the roots (a) and (b) are reduced to

$$\frac{\sqrt{C_1^2 - k^2a^2}}{ka} = \tan ka \quad \text{and} \quad \frac{ka}{\sqrt{C_1^2 - k^2a^2}} = -\tan ka$$

Specifying different values for C_1, we can find the solution by graphical methods. We start by graphing the functions

$$f_1 = \frac{\sqrt{C_1^2 - k^2a^2}}{ka} \quad \text{and} \quad f_2 = \tan ka$$

$$f_3 = \frac{ka}{\sqrt{C_1^2 - k^2a^2}} \quad \text{and} \quad f_4 = -\tan ka$$

Now we can see that for $C_1 < \pi/2$ there is only one energy level, which corresponds to the even wave function. (The curves $y = f_1$ and $y = f_2$ intersect only at one point, and the curves $y = f_3$ and $y = f_4$ do not intersect at all.)

As C_1 increases in magnitude, the number of levels will grow.

46. The Schrödinger equation for a three-dimensional oscillator

$$-\frac{\hbar^2}{2m}\Delta\psi + \left(\frac{k_1x^2}{2} + \frac{k_2y^2}{2} + \frac{k_3z^2}{2}\right)\psi = E\psi \tag{1}$$

permits separating the variables:

$$\psi(x, y, z) = \psi_1(x)\,\psi_2(y)\,\psi_3(z)$$

If we substitute this function into (1) and divide all members by it, we obtain, provided we denote $\omega_i^2 = k_i/m$ $(i = 1, 2, 3)$:

$$\left(-\frac{1}{\psi_1}\frac{\hbar^2}{2m}\frac{d^2\psi_1}{dx^2} + \frac{m\omega_1^2}{2}x^2\right) + \left(-\frac{1}{\psi_2}\frac{\hbar^2}{2m}\frac{d^2\psi_2}{dy^2} + \frac{m\omega_2^2}{2}y^2\right) + \left(-\frac{1}{\psi_3}\frac{\hbar^2}{2m}\frac{d^2\psi_3}{dz^2} + \frac{m\omega_3^2}{2}z^2\right) = E \tag{2}$$

Since x, y, z are independent variables, each parenthesis must be equal to a constant, which we denote E_1, E_2, E_3, respectively. The problem then comes down to three equations for the one-dimensional oscillator

$$-\frac{\hbar^2}{2m}\frac{d^2\psi_1}{dx^2}+\frac{m\omega_1^2}{2}x^2\psi_1=E_1\psi_1, \text{ etc.} \tag{3}$$

and we bear in mind that $E = E_1 + E_2 + E_3$. In dimensionless variables $\xi = x\sqrt{m\omega_1/\hbar}$, $\eta = y\sqrt{m\omega_2/\hbar}$, $\zeta = z\sqrt{m\omega_3/\hbar}$, if we use the known results for the one-dimensional oscillator, we can get the general solution for our case:

$$E_{n_1n_2n_3}=\left(n_1+\frac{1}{2}\right)\hbar\omega_1+\left(n_2+\frac{1}{2}\right)\hbar\omega_2+\left(n_3+\frac{1}{2}\right)\hbar\omega_3$$

$$\psi_{n_1n_2n_3}(\xi,\eta,\zeta)=Ce^{-\frac{\xi^2+\eta^2+\zeta^2}{2}}H_{n_1}(\xi)H_{n_2}(\eta)H_{n_3}(\zeta)$$

Here H_{n_i} are Hermite polynomials, where $n_i = 0, 1, 2, \ldots$ ($i = 1, 2, 3$), and C can be determined from the normalization condition.

Let us normalize $\psi_n=C_ne^{-\xi^2/2}H_n(\xi)$. If we differentiate the identity $\frac{de^{-\xi^2}}{d\xi}+2\xi e^{-\xi^2}=0$ $(n+1)$ times and introduce the definition

$$H_n=(-1)^n e^{\xi^2}\frac{d^ne^{-\xi^2}}{d\xi^n} \tag{4}$$

we see that (4) turns Eq. (5) of Appendix 6 into an identity. Furthermore, it is clear that H_n can be written in the form of a series: $H_n(\xi) = 2^n\xi^n + a_{n-2}\xi^{n-2} + \ldots$. Substituting into the normalization integral

$$\int_{-\infty}^{\infty}|\psi_n(\xi)|^2dx=\sqrt{\frac{\hbar}{m\omega_1}}C_n^2\int_{-\infty}^{\infty}e^{-\xi^2}H_n^2(\xi)\,d\xi=1$$

the definition (4) for one of the Hermite polynomials and integrating n times by parts, we get

$$1=\sqrt{\frac{\hbar}{m\omega_1}}C_n^2(-1)^n\int_{-\infty}^{\infty}\frac{d^ne^{-\xi^2}}{d\xi^n}H_n(\xi)\,d\xi$$

$$=\sqrt{\frac{\hbar}{m\omega_1}}C_n^2\int_{-\infty}^{\infty}e^{-\xi^2}\frac{d^nH_n}{d\xi^n}\,d\xi$$

But since $\frac{d^n H_n}{d\xi^n} = 2^n n!$, it follows that

$$C_n^2 = \sqrt{\frac{m\omega_1}{\pi\hbar}} \frac{1}{2^n n!}$$

It is evident, now, that the normalized solution for the three-dimensional oscillator can be obtained in the form

$$\psi(x, y, z) = \frac{\sqrt[4]{m^3\omega_1\omega_2\omega_3}\, e^{-\frac{\xi^2+\eta^2+\zeta^2}{2}} H_{n_1}(\xi) H_{n_2}(\eta) H_{n_3}(\zeta)}{(\hbar\pi)^{3/4} \times (2^{n_1+n_2+n_3} n_1! \times n_2! \times n_3!)^{1/2}}.$$

47. For this case the Schrödinger equation

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \left(\frac{m\omega^2}{2}x^2 - e|\mathbf{E}|x\right)\psi = E\psi$$

can be reduced to the problem of the harmonic oscillator by completing squares in the expression for the potential energy. Introducing new variables

$$x_1 = x - \frac{e|\mathbf{E}|}{m\omega^2}, \quad \xi = \sqrt{\frac{h}{m\omega}}\, x_1, \quad E_1 = E + \frac{e^2|\mathbf{E}|^2}{2m\omega^2}$$

we arrive at the equation

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx_1^2} + \frac{m\omega^2}{2}x_1^2\psi = E_1\psi$$

and we can (see the solution to Problem 46) write the eigenfunctions

$$\psi_n = C_n e^{-\xi^2/2} H_n(\xi)$$

and eigenvalues

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega - \frac{e^2|\mathbf{E}|^2}{2m\omega^2}$$

of the Hamiltonian.

48. According to the solution of Problem 46 the oscillator in its nth quantum state is characterized by a wave function

$$\psi_n = C_n e^{-\xi^2/2} H_n(\xi)$$

By definition

$$\langle x^2\rangle_n=\int_{-\infty}^{\infty}|\psi_n|^2 x^2\,dx=C_n^2\left(\frac{m\omega}{\hbar}\right)^{-3/2}\int_{-\infty}^{\infty}e^{-\xi^2}H_n^2\xi^2\,d\xi$$

Now we substitute expression (4) from the solution of Problem 46 for one of the Hermite polynomials in the above integral and integrate by parts n times. We get

$$\langle x^2\rangle_n=C_n^2\left(\frac{\hbar}{m\omega}\right)^{3/2}(-1)^n\int_{-\infty}^{\infty}\frac{d^n e^{-\xi^2}}{d\xi^n}H_n\xi^2\,d\xi$$

$$=C_n^2\left(\frac{\hbar}{m\omega}\right)^{\frac{3}{2}}\int_{-\infty}^{\infty}e^{-\xi^2}\frac{d^n(H_n\xi^2)}{d\xi^n}\,d\xi$$

Evidently,

$$\frac{d^n(H_n\xi^2)}{d\xi^n}=\frac{d^n}{d\xi^n}(a_n\xi^{n+2}+a_{n-2}\xi^n+\ldots)=a_n\frac{(n+2)!}{2!}\xi^2+n!\,a_{n-2}$$

Here $a_n=2^n$ and $a_{n-2}=-\frac{n(n-1)\,a_n}{4}$. Next we substitute the values of the following integrals into $\langle x^2\rangle_n$:

$$\int_{-\infty}^{\infty}e^{-\xi^2}\,d\xi=\sqrt{\pi},\qquad\int_{-\infty}^{\infty}\xi^2e^{-\xi^2}\,d\xi=\frac{\sqrt{\pi}}{2}$$

and get

$$\langle x^2\rangle_n=C_n^2\left(\frac{\hbar}{m\omega}\right)^{3/2}\left[2^n\frac{(n+2)!}{2}\frac{\sqrt{\pi}}{2}-2^n\frac{n(n-1)}{4}n!\sqrt{\pi}\right]$$

$$=\frac{\hbar}{m\omega}\left(n+\frac{1}{2}\right)$$

Hence,

$$\langle V\rangle_n=\frac{m\omega^2}{2}\langle x^2\rangle_n=\frac{\hbar\omega}{2}\left(n+\frac{1}{2}\right)=\frac{E_n}{2}.$$

49. Since for the one-dimensional case the kinetic energy operator is $\hat{T}=-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}$, it follows that

$$\langle T\rangle=-\frac{\hbar^2}{2m}\int_{-\infty}^{\infty}\psi^*\frac{d^2\psi}{dx^2}\,dx=\frac{\hbar^2}{2m}\int_{-\infty}^{\infty}\left|\frac{d\psi}{dx}\right|^2dx$$

In the given instance $n=3$ and $\psi=\psi_3=C_3e^{-\xi^2/2}H_3(\xi)$, where $H_3=8\xi^3-12\xi$. We change the variable x to ξ, $x=\xi(m\omega/\hbar)^{-1/2}$, substitute $C_3^2=\sqrt{m\omega/(\hbar\pi)}\frac{1}{2^3\times 3!}$ from the solution of Problem (46), and get

$$\langle T\rangle_3=\frac{\hbar\omega}{2\sqrt{\pi}}\frac{1}{2^3\times 3!}\int\limits_{-\infty}^{\infty}\left\{\frac{d}{d\xi}\left[e^{-\xi^2/2}(8\xi^3-12\xi)\right]\right\}^2 d\xi$$

$$=\frac{\hbar\omega}{3!\sqrt{\pi}}\int\limits_{-\infty}^{\infty}e^{-\xi^2}(4\xi^8-36\xi^6+93\xi^4-54\xi^2+9)\,d\xi$$

The integrals we have here are done easily if we differentiate with respect to a parameter. Since $\int\limits_{-\infty}^{\infty}e^{-\alpha\xi^2}d\xi=\sqrt{\pi/\alpha}$,

$$\int\limits_{-\infty}^{\infty}\xi^{2n}e^{-\alpha\xi^2}d\xi=\left(-\frac{\partial}{\partial\alpha}\right)^n\int\limits_{-\infty}^{\infty}e^{-\alpha\xi^2}d\xi$$

$$=\frac{1\times 3\times\ldots\times(2n-1)}{2^n}\sqrt{\frac{\pi}{\alpha^{2n+1}}}$$

and

$$\langle T\rangle_3=\frac{\hbar\omega}{3!\sqrt{\pi}}\times\frac{21}{2}\times\sqrt{\pi}=\frac{1}{2}\hbar\omega\left(3+\frac{1}{2}\right).$$

50. To solve the equation

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2}+V_0(e^{-2\alpha x}-2e^{-\alpha x})\psi=E\psi$$

it is convenient to introduce a new variable $y=e^{-\alpha x}$. We denote

$$\beta^2=\frac{2mV_0}{\hbar^2\alpha^2} \tag{1}$$

and

$$\varepsilon=\frac{2mE}{\hbar^2\alpha^2} \tag{2}$$

and we get an equation of the form

$$y^2\frac{d^2\psi}{dy^2}+y\frac{d\psi}{dy}-\beta^2(y^2-2y)\psi+\varepsilon\psi=0$$

Let us first study it at points

$$y \to \infty \ (x \to -\infty) \text{ and } y \to 0 \ (x \to \infty)$$

In the first case the equation takes the form

$$\frac{d^2\psi_\infty}{dy^2} - \beta^2\psi_\infty = 0$$

This has a solution

$$\psi_\infty = e^{-\beta y}$$

(The second particular solution, $\psi_\infty = e^{+\beta y}$, will tend to infinity when y tends to infinity.) Near point $y = 0$ the substitution of $\psi = y^k$ gives an equation for k:

$$k^2 + \varepsilon = 0, \quad \text{or} \quad k = \pm\sqrt{-\varepsilon}$$

where we leave none but the terms of the lowest order.

For $\varepsilon > 0$ both solutions will serve ($\psi_0 = y^{\pm\sqrt{-\varepsilon}} = e^{\pm\alpha\sqrt{-\varepsilon}\,x}$ remains finite if $\sqrt{-\varepsilon}$ is imaginary); the energy spectrum is continuous.

For $\varepsilon = -\lambda^2 < 0$ only the solution $\psi_0 = y^\lambda \ (\lambda > 0)$ satisfies the requirement of finiteness.

To find the general solution, we must substitute $\psi = y^\lambda e^{-\beta y} F(y)$ and then solve the obtained equation for $F(y)$:

$$y^2\frac{d^2F}{dy^2} + [(1 + 2\lambda)\, y - 2\beta y^2]\frac{dF}{dy} + 2\beta y\left[\beta - \lambda - \frac{1}{2}\right]F = 0$$

Next we express $F(y)$ as a power series, $F(y) = \sum\limits_{k=0}^{\infty} a_k y^k$, and when this is inserted into the equation, we find that the coefficients of y^k satisfy the recurrence relation

$$a_{k+1} = a_k \frac{2\beta\left(k + \lambda + \frac{1}{2} - \beta\right)}{(k+1)(k+1+2\lambda)} \quad (k = 0, 1, 2, \ldots)$$

Since for large k's the relation becomes $\frac{a_{k+1}}{a_k} \approx \frac{2\beta}{k}$ and, hence, $F_{y\to\infty} \approx e^{2\beta y}$, the only solution that will satisfy the requirement of finiteness as $y \to \infty$ is the one represented by a cut-off series. We can obtain this if for $k = n$, say, $\beta - n - 1/2 = \lambda$; λ must be nonnegative. Now we substitute β and λ from (1) and (2) into this condition, and we get the discrete energy spectrum $E_n = -V_0 \times$

$$\times\left[1-\sqrt{\frac{\hbar^2\alpha^2}{2mV_0}}\left(n+\frac{1}{2}\right)\right]^2 \quad \text{with} \quad n=0,\ 1,\ 2,\ \ldots.$$

The number of levels is restricted by the condition

$$n+\frac{1}{2}<\beta=\sqrt{\frac{2mV_0}{\hbar^2\alpha^2}}$$

and, apparently, depends on the depth V_0 of the well. For instance, there are no discrete levels for $\beta < 1/2$.

51. The equation $-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2}-\frac{e^2}{|x|}\psi=E\psi$ for $E>0$ will have a continuous spectrum. Let us consider the case of $E<0$. We denote $\gamma^2=-\frac{2mE}{\hbar^2}$ and $\varkappa=\frac{me^2}{\hbar^2\gamma}$ and introduce the new variable $\xi=2\gamma x$. Now we can write the equation in dimensionless variables:

$$\frac{d^2\psi}{d\xi^2}+\left(\frac{\varkappa}{|\xi|}-\frac{1}{4}\right)\psi=0 \tag{1}$$

First let us consider the region $\xi>0$.

When ξ tends to infinity, the equation transforms into $\frac{d^2\psi_\infty}{d\xi^2}-\frac{1}{4}\psi_\infty=0$ with a solution that satisfies the condition of finiteness, namely $\psi_\infty=e^{-\xi/2}$.

We introduce a new function $f(\xi)$ and substitute $\psi = e^{-\xi/2}f(\xi)$ into Eq. (1). We thus get an equation for $f(\xi)$: $\xi\frac{d^2f}{d\xi^2}-\xi\frac{df}{d\xi}+\varkappa f=0$. To study the behaviour of $f(\xi)$ near $\xi=0$ we substitute ξ^α for f and, retaining the members of the lowest order, we find that $\alpha(\alpha-1)=0$. In other words, there are two solutions: (1) $f(\xi)=1$ and (2) $f(\xi)=\xi$. The first solution cannot be used, however, since the finiteness of $\psi(\xi)$ at point $\xi=0$ will mean that $V\psi=-\frac{e^2\psi}{\xi}\to\infty$ and, hence, $\frac{d^2\psi}{d\xi^2}\to\infty$. Thus, $\psi(\xi)$ will also be infinite. For this reason, none but the second solution is suitable.

The substitution of $f(\xi)=\sum_{k=1}^{\infty}a_k\xi^k$ leads in the usual way to a recurrence relation for the coefficients:

$$a_{k+1}=a_k\frac{k-\varkappa}{(k+1)k} \quad (k=1,\ 2,\ \ldots)$$

Next we take the limit of this formula for $k \gg 1$, and we find that in this case $f(\xi)$ turns into e^{ξ}, since their series expansions coincide. Hence, for $\psi = e^{-\xi/2} f(\xi)$ to be finite, $f(\xi)$ must become a polynomial, and this is possible when its series expansion is cut off at the nth member.

We can see from the recurrence relation that $\varkappa$ must equal n for $a_{n+1} = 0$. If we substitute $n = \varkappa = \frac{me^2}{\hbar^2} \times \frac{\hbar}{\sqrt{-2mE}}$, we get a discrete energy spectrum

$$E_n = -\frac{me^4}{2\hbar^2 n^2}$$

and the corresponding wave functions are

$$\psi_n(\xi) = e^{-\xi/2}\xi \sum_{k=0}^{n-1} a_{k+1}\xi^k$$

To find the solution for $\xi < 0$ we introduce $\eta = -\xi$. The equation takes on the very same form as for $\xi > 0$:

$$\frac{d^2\psi(\eta)}{d\eta^2} + \left(\frac{\varkappa}{\eta} - \frac{1}{4}\right)\psi = 0$$

and its solution, which continues $\psi(\xi)$ into the region $\xi > 0$, is

$$\psi_n(\eta) = -e^{-\eta/2}\eta \sum_{k=0}^{n-i} a_{k+1}\eta^k = e^{\xi/2}\xi \sum_{k=0}^{n-1} a_{k+1}(-\xi)^k.$$

52. To find the wave function for the spherically symmetric oscillator, as for any problem with a central symmetry, we must look for it in the form

$$\psi(r, \theta, \varphi) = f(r)\, P_{lm}(\cos\theta)\, e^{im\varphi}$$

where $P_{lm}(\cos\theta)$ is the Legendre polynomial $l = 0, 1, 2, \ldots$ and $m = 0, \pm 1, \ldots, \pm l$.

Then $f(r)$ must satisfy the equation

$$-\frac{\hbar^2}{2\mu}\left[\frac{d^2 f}{dr^2} + \frac{2}{r}\frac{df}{dr} - \frac{l(l+1)}{r^2} f\right] + \frac{\mu\omega^2}{2} r^2 f = Ef$$

If we introduce the new function $U = rf$ and, as in the solution to Problem 46, proceed to $\xi = r\sqrt{\frac{\mu\omega}{\hbar}}$ and $\lambda = \frac{E}{\hbar\omega}$,

the equation takes the form

$$\frac{d^2U}{d\xi^2}+\left[2\lambda-\frac{l(l+1)}{\xi^2}-\xi^2\right]U=0$$

Obviously, as ξ tends to infinity, $U_\infty = e^{-\xi^2/2}$. And as ξ tends to zero, if we substitute $U_0 = \xi^\alpha$ for α, we get the equation $\alpha(\alpha - 1) = l(l + 1)$. Hence, $\alpha_1 = l + 1$ and $\alpha_2 = -l$, and none but α_1 gives the finite function as ξ tends to zero, i.e. $U_0 = \xi^{l+1}$.

We use these results to find $U(\xi)$ in the form

$$U=e^{-\xi^2/2}\xi^{l+1}v(\xi)$$

After substituting, we get the equation for $v(\xi)$:

$$\frac{d^2v}{d\xi^2}+2\frac{dv}{d\xi}\left[\frac{l+1}{\xi}-\xi\right]+2\left[\lambda-l-\frac{3}{2}\right]v=0$$

and we look for the solution in the form of a series $v = \sum_{k=k_0}^{\infty} a_k\xi^k$. If we equate the coefficients of each power of ξ with zero, we get a recurrence relation for the coefficients

$$\underset{(k\geqslant k_0)}{a_{k+2}}=a_k\frac{2\left[k+l+\frac{3}{2}-\lambda\right]}{(k+2)(k+2l+3)}\approx\frac{2a_k}{k}\ (\text{as } k\to\infty) \qquad (1)$$

Next we determine the lowest exponent k_0. It follows from $k_0(k_0 + 1) - 2(l + 1)k_0 = 0$ that none but the solutions with $k_0 = 0$ remain finite at zero. From formula (1) it follows that as $\xi \to \infty$ this series turns into e^{ξ^2} and $U \to e^{\xi^2/2}$. To avoid this we must cut off the series, which can only be done when $\lambda = p + l + \frac{3}{2}$. If we also consider that the series can only begin with $k_0 = 0$, then in this expression p must be even, i.e.

$$p = 2n, \text{ where } n = 0, 1, 2, \ldots$$

Then

$$\lambda=2n+l+\frac{3}{2} \quad\text{and}\quad E_n=\hbar\omega\left(2n+l+\frac{3}{2}\right)$$

The function that corresponds to an energy level with given n and l is

$$\psi_{nlm}(r,\theta,\varphi)=C\frac{e^{-\xi^2/2}\xi^{l+1}v_n(\xi)}{\xi}P_{lm}(\cos\theta)e^{im\varphi}$$

Obviously, the level is degenerate: when $N = 2n + l$ is even, we get $l = 0, 2, 4, \ldots, N$; when N is odd, we get $l = 1, 3, 5, \ldots, N$. In addition, for given n and l the level is $(2l + 1)$-fold degenerate in m.

53. For the two-dimensional problem,

$$\Delta = \frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2}$$

If $V = V(\rho)$ and does not depend on angle φ, the variables can be separated, which means that the wave function can be sought in the form

$$\psi(\rho, \varphi) = e^{im\varphi} U(\rho)$$

($\hat{H}$ and $\hat{L}_z$ commute in such a field.) For U we get the equation

$$-\frac{\hbar^2}{2\mu}\left(\frac{d^2U}{d\rho^2} + \frac{1}{\rho}\frac{dU}{d\rho} - \frac{m^2}{\rho^2}U\right) - \frac{Ze^2}{\rho}U = EU$$

Now we consider $E < 0$ (for $E > 0$ the spectrum is continuous). Substituting at for ρ and introducing the new function $f = \sqrt{\rho}U$, we get the equation

$$\frac{d^2f}{dt^2} + \left(\frac{2\mu E}{\hbar^2}a^2 + \frac{2\mu Ze^2}{\hbar^2}\frac{a}{t} - \frac{m^2 - 1/4}{t^2}\right)f = 0$$

We choose $a = \frac{\hbar^2}{\mu Ze^2}$ and denote

$$\gamma^2 = -\frac{2\mu E}{\hbar^2}a^2 \tag{1}$$

We thereby simplify this equation and get

$$\frac{d^2f}{dt^2} + \left(\frac{2}{t} - \gamma^2 - \frac{m^2 - \frac{1}{4}}{t^2}\right)f = 0$$

$$(0 \leqslant t \leqslant \infty)$$

As $t \to \infty$, we find that $f_\infty = e^{-\gamma t}$, and we discard the solution $e^{+\gamma t}$.

For $E > 0$, it follows from (1) that $\gamma = i\beta$. Then both solutions $e^{\pm\gamma t} = e^{\pm i\beta\rho/a}$ are finite and the spectrum of E is continuous.

As $t \to 0$, assuming that $f_0 = t^\alpha$, we get $\alpha(\alpha-1) = m^2 - \frac{1}{4}$, i.e. $\alpha_1 = m + \frac{1}{2}$ and $\alpha_2 = -\left(m - \frac{1}{2}\right)$. Since $m = 0, \pm 1, \pm 2, \ldots$, we must choose $\alpha = |m| + \frac{1}{2}$ and look for the general solution in the form

$$f(t) = e^{-\gamma t} t^{|m|+\frac{1}{2}} v(t)$$

As a result, for the function $v(t)$ we get the equation

$$t\frac{d^2v}{dt^2} + 2\left[|m| + \frac{1}{2} - \gamma t\right]\frac{dv}{dt} + 2\left[1 - \left(|m| + \frac{1}{2}\right)\gamma\right]v = 0$$

and look for the solution in the form of a series $v = \sum\limits_{k=k_0}^{\infty} a_k t^k$. If we equate the coefficient of the lowest power of t with zero, i.e. the coefficient of t^{k_0-1}, we get

$$k_0(k_0 + 2|m|) = 0$$

whence

$$k_0 = 0 \text{ or } k_0 = -2|m|$$

The second solution for $t \to 0$ gives a function that tends to infinity. The coefficients of t^k, where $k = 0, 1, \ldots$, being equated with zero, give the recurrence relation

$$a_{k+1} = a_k \frac{2\left[\gamma\left(k + |m| + \frac{1}{2}\right) - 1\right]}{(k+1)(k+1+2|m|)} \tag{2}$$

From this for $k \gg 1$ we get $\frac{a_{k+1}}{a_k} \to \frac{2\gamma}{k}$. (The function $e^{2\gamma t}$, when expanded into a series, satisfies the same relation for its coefficients.) To prevent $v(t)$ from turning into $e^{2\gamma t}$ as t grows, we must cut off the series at, say, $k = n$, i.e. so that

$$\gamma\left[n + |m| + \frac{1}{2}\right] - 1 = 0$$

[see formula (2)]. According to (1) we find that for the two-dimensional Kepler problem the energy of a particle takes

on the following values:

$$E_n = -\frac{\mu e^4 Z^2}{2\hbar^2\left(n+|m|+\frac{1}{2}\right)^2}$$

$$(n = 0, 1, 2, \ldots, \ |m| = 0, 1, 2, \ldots)$$

The function corresponding to an energy level with given n and m is

$$\psi_{nm}(\rho, \varphi) = e^{im\varphi}\frac{1}{\sqrt{\rho}}e^{-\nu\frac{\rho}{a}}\left(\frac{\rho}{a}\right)^{|m|+\frac{1}{2}}\sum_{k=0}^{n} a_k\left(\frac{\rho}{a}\right)^k.$$

54. For a particle in a central field,

$$\psi(r, \theta, \varphi) = e^{im\varphi}\ P_{lm}(\cos\theta)\, f(r)$$

Its radial part $f(r)$ satisfies the equation

$$\frac{d^2 f}{dr^2}+\frac{2}{r}\frac{df}{dr}+\left(\frac{2\mu E}{\hbar^2}-\frac{l(l+1)}{r^2}\right)f=0$$

for $r<R$, and $f=0$ for $r>R$ (see the solution to Problem 44). Hence, for f which is the solution for the above equation the boundary condition will be $f(R) = 0$.

If we introduce $k^2=\frac{2\mu E}{\hbar^2}$ and $\chi(r)=(r)^{1/2} f(r)$, for $\chi(r)$ we get the Bessel equation

$$\frac{d^2\chi}{dr^2}+\frac{1}{r}\frac{d\chi}{dr}+\left[k^2-\frac{\left(l+\frac{1}{2}\right)^2}{r^2}\right]\chi=0$$

Since $\chi(r)=J_{\pm\left(l+\frac{1}{2}\right)}(kr)\to r^{\pm\left(l+\frac{1}{2}\right)}$ as $r\to 0$, the only solution that satisfies the requirement of finiteness is

$$J_{l+\frac{1}{2}}(kr)\to\frac{a}{\sqrt{r}}\sin kr \quad (\text{if } l=0)$$

The energy levels corresponding to these functions can be derived from the continuity conditions for the function at $r=R$, i.e. $J_{l+\frac{1}{2}}(kR) = 0$. We denote the roots of this Bessel function by $b_n^{(l)}$ and write

$$E_n^{(l)}=\frac{\hbar^2(b_n^{(l)})^2}{2\mu R^2}$$

Evidently, at $l=0$,

$$b_n^{(0)}=n\pi \quad \text{and} \quad E_n^{(0)}=\frac{\hbar^2n^2\pi^2}{2\mu R^2}.$$

55. When the electron in the hydrogen atom is on the lowest energy level, then $n=1$, $l=0$, and $m=0$. According to formula (III-24), the function that corresponds to this level is $\psi_{100}=Ce^{-r/a}$. The constant C is determined from the normalization condition

$$\int|\psi_{100}|^2\,d\tau=4\pi C^2\int_0^\infty e^{-2r/a}r^2\,dr=\pi C^2a^3=1$$

The integral is of the type $\int_0^\infty e^{-x}x^n dx=n!$. Then $|\psi_{100}|^2d\tau$ is the probability of finding the electron in $d\tau$. We then calculate

$$\langle r\rangle=\int_0^\infty r|\psi_{100}|^2\,4\pi r^2\,dr=\frac{4}{a^3}\int_0^\infty e^{-2r/a}r^3\,dr=\frac{4}{a^3}\frac{3!}{2^4}a^4=\frac{3}{2}a$$

$$\langle r^2\rangle=\int_0^\infty r^2|\psi_{100}|^2\,4\pi r^2\,dr=\frac{4}{a^3}\int_0^\infty e^{-2r/a}r^4\,dr=3a^2$$

We find the most probable value r_0 by equating with zero the derivative of $|\psi_{100}|^2r^2=C^2e^{-2r/a}r^2$, which determines the probability of finding the electron at a given distance from the origin. Hence, $r_0=a$.

56. To write the required functions, we use formulas (III-23), (III-24), and (III-24a). At $n=2$, l can be equal to 0 or 1.

If $l=0$, the sum in formula (III-24) has two members, with $k=0$ and $k=1$, and the angular part $Y_{00}(\theta,\varphi)=1$. Thus $\psi_{200}(r,\theta,\varphi)=b_0e^{-r/(2a)}\left(1+\frac{b_1}{b_0}\frac{r}{a}\right)$, where $\frac{b_1}{b_0}=-\frac{1}{2}$ [see formula (III-24a)], and the normalization condition

$$1=\int\int\int|\psi_{200}|^2r^2\,dr\sin\theta\,d\theta\,d\varphi=b_0^2 8\pi a^3$$

yields

$$b_0=\frac{1}{\sqrt{8\pi a^3}}$$

If $l = 1$, there is only one member in the sum, $k = 1$, and the functions are

$$\psi_{210} = b_1 e^{-r/(2a)} \frac{r}{a} \cos\theta$$

$$\psi_{211} = b_1' e^{-r/(2a)} \frac{r}{a} \sin\theta\, e^{i\varphi}$$

$$\psi_{21-1} = b_1' e^{-r/(2a)} \frac{r}{a} \sin\theta\, e^{-i\varphi}$$

The normalization constants are

$$b_1 = \frac{1}{2\sqrt{8\pi a^3}} \quad \text{and} \quad b_1' = \frac{1}{8\sqrt{\pi a^3}}.$$

57. Since the potential energy does not depend on the angles

$$\psi(r, \theta, \varphi) = U(r)\, P_{lm}(\cos\theta)\, e^{im\varphi}$$

and the radial part of the function, $U(r)$, satisfies the equation

$$-\frac{\hbar^2}{2\mu}\left(\frac{d^2U}{dr^2} + \frac{2}{r}\frac{dU}{dr} - \frac{l(l+1)U}{r^2}\right) - \left(\frac{e^2}{r} - \frac{C}{r^2}\right)U = EU \qquad (1)$$

Next we turn to the dimensionless coordinate $\rho = r/a$, where $a = \frac{\hbar^2}{\mu e^2}$, and energy $\varepsilon = \frac{2E\hbar^2}{\mu e^4} = -\gamma^2$ (we will examine none but the discrete energy spectrum $E < 0$). We collect in formula (1) the members of type $1/r^2$ and denote $s(s+1) = l(l+1) + \frac{2\mu C}{\hbar^2}$. Now we can rewrite the equation in the form

$$\frac{d^2U}{d\rho^2} + \frac{2}{\rho}\frac{dU}{d\rho} - \frac{s(s+1)}{\rho^2}U + \left(\frac{2}{\rho} - \gamma^2\right)U = 0$$

We introduce the function $\chi = rU$ and get the equation

$$\frac{d^2\chi}{d\rho^2} + \left[\frac{2}{\rho} - \gamma^2 - \frac{s(s+1)}{\rho^2}\right]\chi = 0$$

Using the usual method, we find $\chi_\infty = e^{-\gamma\rho}$ and $\chi_0 = \rho^{s+1}$ (the second solution $\chi_0 = \rho^{-s}$ is not suitable for $s > 0$). The substitution of $\chi(\rho) = e^{-\gamma\rho}\rho^{s+1}f$ gives the equation for f:

$$\rho\frac{d^2f}{d\rho^2} + [2(s+1) - 2\gamma\rho]\frac{df}{d\rho} - 2[\gamma(s+1) - 1]f = 0$$

and we look for the solution in the form of a power series

$$f=\sum_{k=0}^{\infty} a_k \rho^k$$

By equating the coefficients of ρ^k with zero, we get

$$a_{k+1}=a_k \frac{2[\gamma(k+s+1)-1]}{(k+1)(k+2s+2)} \quad (k=0, 1, 2, \ldots)$$

To prevent f from turning into $e^{2\gamma\rho}$ as $\rho \to \infty$ $\left(\text{the possibility can be seen from the asymptotic relation } \frac{a_{k+1}}{a_k} \underset{k\gg 1}{=} \frac{2\gamma}{k}\right)$ we must cut off the series, i.e. we must see that

$$\gamma=\sqrt{-\frac{2\hbar^2 E}{\mu e^4}}=\frac{1}{k+s+1}$$

Hence, the energy levels of the particle are determined by the formula

$$E_{ks}=-\frac{\mu e^4}{2\hbar^2 (k+s+1)^2}$$

where $k=0, 1, \ldots$, and $s=-\frac{1}{2}+\sqrt{\left(l+\frac{1}{2}\right)^2+\frac{2\mu C}{\hbar^2}} \approx$ $\approx l+\frac{\mu C}{\hbar^2\left(l+\frac{1}{2}\right)}$ if $\frac{2\mu C}{\hbar^2} \ll l+\frac{1}{2}$.

If we introduce the principal quantum number $n=k+$ $+l+1$, the energy

$$E_{nl}=\frac{-\mu e^4}{2\hbar^2\left[n+\frac{\mu C}{\hbar^2\left(l+\frac{1}{2}\right)}\right]^2}$$

proves dependent on n and l, and the functions that correspond to these levels

$$\psi_{nlm}(r, \theta, \varphi)=e^{-\gamma\rho}\rho^s \sum_{p=0}^{k} a_p \rho^p P_{lm}(\cos\theta)\, e^{im\varphi}$$

depend on the numbers n, l, and m. We now see that E_{nl} is $(2l+1)$-fold degenerate (since $m=0, \pm 1, \ldots, \pm l$).

58. As usual we assume that $\psi(r, \theta, \varphi)=U(r)\, P_{lm}(\cos\theta) \times$ $\times e^{im\varphi}$, and for the function $F=rU$ we get a problem

similar to Problem 52. Its solution gives us

$$E_{nl}=\hbar\sqrt{\frac{B}{2\mu}}\left[4n+2+\sqrt{(2l+1)^2+\frac{8\mu A}{\hbar^2}}\right]$$

$$U=\frac{e^{-\xi/2}}{\sqrt{\xi}}\xi^{\alpha}\sum_{k=0}^{n}a_k\xi^k$$

where

$$\xi=\frac{\sqrt{2\mu B}}{\hbar}r^2$$

$$a_{k+1}=a_k\frac{k-n}{(k+1)\left(k+2\alpha+\frac{1}{2}\right)}\quad(k=0,\ 1,\ 2,\ \ldots)$$

$$\alpha=\frac{1}{4}+\frac{1}{4}\sqrt{(2l+1)^2+\frac{8\mu A}{\hbar^2}}.$$

59. Using the solutions to Problems 43 and 46, we can write the solution in the form

$$\psi_{nn_1n_2}(x,\ y,\ z)=Ae^{-\xi^2/2}H_n(\xi)\sin\frac{n_1\pi y}{a}\sin\frac{n_2\pi z}{b}$$

$$\xi=x\sqrt{\frac{\mu\omega}{\hbar}}$$

for $0\leqslant y\leqslant a$ and $0\leqslant z\leqslant b$. In the rest of the space $\psi=0$. The energy levels that correspond to the function are

$$E_{nn_1n_2}=\hbar\omega\left(n+\frac{1}{2}\right)+\frac{\hbar^2n_1^2\pi^2}{8\mu a^2}+\frac{\hbar^2n_2^2\pi^2}{8\mu b^2}$$

The normalization condition yields

$$A^2=\frac{4}{ab}\sqrt{\frac{\mu\omega}{\hbar\pi}}\times\frac{1}{2^n n!}.$$

60. We denote the ordinary coordinate as r', and the Laplacian in relation to r' as Δ'. Next we turn to dimensionless variables $r=r'/a$ and $\Delta=a^2\Delta'$ (where $a=\hbar^2/\mu e^2$ is the Bohr radius), and to the dimensionless energy ε defined by the relation $E=\varepsilon\frac{e^2}{a}=\varepsilon\frac{\mu e^4}{\hbar^2}$, and we write the Schrödinger equation for the hydrogen atom:

$$-\frac{1}{2}\Delta\psi-\frac{1}{r}\psi=\varepsilon\psi$$

In parabolic coordinates (see Appendix 3) the equation takes the form

$$\frac{2}{u+v}\left[\frac{\partial}{\partial u}\left(u\frac{\partial\psi}{\partial u}\right)+\frac{\partial}{\partial v}\left(v\frac{\partial\psi}{\partial v}\right)\right.$$
$$\left.+\frac{1}{4}\left(\frac{1}{u}+\frac{1}{v}\right)\frac{\partial^2\psi}{\partial\varphi^2}\right]+\frac{2}{u+v}\psi+\varepsilon\psi=0$$

Evidently, we can separate the variables:

$$\psi(u, v, \varphi) = U(u)\,V(v)\,\Phi(\varphi)$$

Here $\Phi(\varphi)$ satisfies the equation

$$\frac{1}{\Phi}\frac{d^2\Phi}{d\varphi^2}=\text{constant}=-m^2$$

whence

$$\Phi = e^{im\varphi}$$

where $m = 0, \pm 1, \pm 2, \ldots$.

We substitute $\frac{\partial^2\psi}{\partial\varphi^2} = -m^2\psi$ into the equation and multiply it by $\frac{u+v}{2}$. We thereby get like equations for $U(u)$ and $V(v)$:

$$\frac{d}{du}\left(u\frac{dU}{du}\right)-\frac{m^2}{4u}U+\frac{\varepsilon}{2}uU+\alpha U=0 \tag{1}$$

$$\frac{d}{dv}\left(v\frac{dV}{dv}\right)-\frac{m^2}{4v}V+\frac{\varepsilon}{2}vV+\beta V=0 \tag{2}$$

with

$$\alpha+\beta=1 \tag{3}$$

We seek the solution of (1) by examining $U(u)$ as $u \to \infty$ and as $u \to 0$. Obviously, U_∞ is determined from the equation

$$\frac{d^2U_\infty}{du^2}+\frac{\varepsilon}{2}U_\infty=0, \quad \text{i.e.} \quad U_\infty=e^{\pm u\sqrt{-\frac{\varepsilon}{2}}} \quad (u\geqslant 0)$$

For $\varepsilon > 0$ both solutions are finite and the energy spectrum is continuous. For $\varepsilon < 0$ the only finite solution is $U_\infty = \exp(-u\sqrt{\varepsilon/2})$. Let us find the energy spectrum $\varepsilon < 0$.

For $u \to 0$ we assume that $U_0 = u^\nu$, and by equating with zero the coefficient of the lowest power of u, we get

$\gamma^2 - \frac{m^2}{4} = 0$ or $\gamma = \pm m/2$. We must put $\gamma = |m|/2$ for the finiteness of U_0.

We look for the general solution in the form

$$U = u^{|m|/2} e^{-u\sqrt{-\frac{\varepsilon}{2}}} \times F(u), \quad \text{where} \quad F = \sum_{k=0}^{\infty} a_k u^k$$

The equation for $F(u)$ in the form

$$u\frac{d^2F}{du^2} + \left(|m| + 1 - 2u\sqrt{-\frac{\varepsilon}{2}}\right)\frac{dF}{du} + \left[\alpha - (|m|+1)\sqrt{-\frac{\varepsilon}{2}}\right]F = 0$$

gives us the recurrence relation for a_k, which we get by equating with zero the coefficient of u^k ($k = 0, 1, 2, \ldots$):

$$a_{k+1} = a_k \frac{\sqrt{-\frac{\varepsilon}{2}}(2k + |m| + 1) - \alpha}{(k+1)(k+|m|+1)} \approx a_k \frac{2\sqrt{-\frac{\varepsilon}{2}}}{k} \quad \text{for} \quad k \gg 1$$

For large u's, i.e. when the members with $k \gg 1$ play the dominant role in F, we find the asymptotic form $F(u) \to \to \exp\left(2u\sqrt{-\varepsilon/2}\right)$ $\left[\text{since } e^{px} = \sum_{k=0}^{\infty} \frac{(px)^k}{k!}\right.$ and the ratio of the coefficients of x^{k+1} and x^k in this series is $\left.\frac{p^{k+1}}{(k+1)!} : \frac{p^k}{k!} \approx \frac{p}{k}\right]$. Hence,

$$U = e^{-u\sqrt{-\frac{\varepsilon}{2}}} \times F \times u^{|m|/2} \to e^{+u\sqrt{-\frac{\varepsilon}{2}}} \times u^{|m|/2}$$

and tends to infinity as $u \to \infty$.

Thus, the series for $F(u)$ must be cut off. This is possible if for, say, $k = n_1$, the following equality holds:

$$\alpha = \sqrt{-\frac{\varepsilon}{2}}(2n_1 + |m| + 1)$$

Then $F(u) = \sum_{k=0}^{n_1} a_k u^k = F_{n_1}(u)$ becomes a polynomial of degree n_1, and U is finite everywhere.

We solve Eq. (2) in a similar manner and we find that for $V(v)$

$$\beta=\sqrt{-\frac{\varepsilon}{2}}\,(2n_2+|m|+1)$$

and

$$V_{n_2}=e^{-v\sqrt{-\frac{\varepsilon}{2}}}\times v^{|m|/2}\times F_{n_2}(v)$$

It follows from condition (3) that

$$2\,(n_1+n_2+|m|+1)\sqrt{-\frac{\varepsilon}{2}}=1$$

Introducing the principal quantum number $n=n_1+n_2+|m|+1$, which obviously can take on the values $n=1, 2, \ldots,$ we get

$$\varepsilon_n=-\frac{1}{2n^2}\quad\text{and}\quad \psi_{n_1n_2m}(u,\,v,\,\varphi)=CU_{n_1}(u)V_{n_2}(v)e^{im\varphi}$$

$$=Ce^{-\frac{u+v}{2n}}u^{|m|/2}v^{|m|/2}F_{n_1}(u)F_{n_2}(v)e^{im\varphi}.$$

61. In a central field for $l=0$ the wave function is simply the radial part, $\psi=R(r)$, which satisfies the equation

$$-\frac{\hbar^2}{2\mu}\left[\frac{1}{r}\frac{d^2}{dr^2}(rR)\right]-V_0e^{-r/a}R=ER$$

We will confine ourselves to $E<0$. We introduce the new function $\chi=rR$ and make a change of variables $y=e^{-r/2a}$. Then χ will satisfy the equation

$$\frac{d^2\chi}{dy^2}+\frac{1}{y}\frac{d\chi}{dy}+\left(C^2-\frac{q^2}{y^2}\right)\chi=0$$

where

$$C^2=\frac{8\mu V_0a^2}{\hbar^2},\quad q^2=-\frac{8\mu Ea^2}{\hbar^2}>0$$

This is Bessel's differential equation whose general solution is

$$\chi=C_1J_q(Cy)+C_2J_{-q}(Cy)$$

As $y\to 0$ (i.e. as $r\to\infty$), χ must remain finite. In this case $J_{-q}\propto y^{-q}$ becomes infinite. Hence, $C_2=0$. As $y\to 1$ (as $r\to 0$), χ must be equal to zero. Therefore, $J_q(C)=0$,

i.e. any value of C must be a root of the qth Bessel function. Since the values of the roots increase as q increases and the first root of J_0 is approximately 2.4, we can estimate $C > 2.4$, i.e.

$$a^2V_0 > \frac{\hbar^2}{8\mu} \times (2.4)^2 = 0.72\,\frac{\hbar^2}{\mu}$$

This is the condition for there to be at least one energy level in a potential well with a depth of the order of V_0 and a width of the order of a. For $r > a$ and as $y \to 0$, we can obtain a similar form of the wave function

$$J_q(Cy) = Ay^q = A\exp\left(-\frac{rq}{2a}\right) = A\exp\left(-\frac{r\sqrt{-2\mu E}}{\hbar}\right).$$

62. First we link the solutions $\psi(x)$ and $\psi(x+l)$. The Schrödinger equations for these cases are

$$-\frac{\hbar^2}{2\mu}\frac{d^2\psi(x)}{dx^2} + V(x)\,\psi(x) = E\psi(x)$$

$$-\frac{\hbar^2}{2\mu}\frac{d^2\psi(x+l)}{d(x+l)^2} + V(x+l)\,\psi(x+l) = E\psi(x+l)$$

We see that because $V(x+l) = V(x)$ and $\frac{d^2}{d(x+l)^2} = \frac{d^2}{dx^2}$, both functions correspond to one energy level E. Considering E a simple eigenvalue, we find that these functions differ only in a constant factor, i.e.

$$\psi(x+l) = \rho\psi(x)$$

In the general case we can obviously find that

$$\psi(x+nl) = \rho^n\psi(x) \quad (n = 0, \pm 1, \ldots) \tag{1}$$

It follows then from the requirement of the finiteness of $\psi(x)$ that $|\rho| = 1$, i.e. $\rho = e^{ikl}$, where k is any real number;

$$\psi(x+l) = e^{ikl}\psi(x) \tag{2}$$

If we solve the Schrödinger equation for $-b \leqslant x \leqslant 0$ (the first region), for $0 \leqslant x \leqslant a$ (the second region), and for $a \leqslant x \leqslant b$ (the third region), and use condition (2) we obviously will have a solution for all x's.

First region ($V = V_0$). The equation takes the form

$$-\frac{\hbar^2}{2\mu}\frac{d^2\psi_{\mathrm{I}}}{dx^2} + V_0\psi_{\mathrm{I}} = E\psi_{\mathrm{I}}$$

Denoting $\lambda^2 = \frac{2\mu(V_0 - E)}{\hbar^2}$, we can write

$$\psi_{\mathrm{I}} = C_1 e^{\lambda x} + C_2 e^{-\lambda x}$$

Second region ($V = 0$). Here the equation is

$$-\frac{\hbar^2}{2\mu}\frac{d^2\psi_{\mathrm{II}}}{dx^2} = E\psi_{\mathrm{II}}$$

and the solution is

$$\psi_{\mathrm{II}} = C_3 e^{i\varkappa x} + C_4 e^{-i\varkappa x}$$

where $\varkappa^2 = 2\mu E/\hbar^2$.

Third region ($V = V_0$). The equation is the same as in the first region, i.e.

$$-\frac{\hbar^2}{2\mu}\frac{d^2\psi_{\mathrm{III}}}{dx^2} + V_0\psi_{\mathrm{III}} = E\psi_{\mathrm{III}}$$

and its solution is

$$\psi_{\mathrm{III}} = C_5 e^{\lambda x} + C_6 e^{-\lambda x}$$

If x lies in the first region, $x + l$ lies in the third, and according to formula (2) the solutions are linked by the condition

$$\psi_{\mathrm{III}}(x + l) = e^{ikl}\psi_{\mathrm{I}}(x)$$

Hence, $C_5 = C_1 e^{ikl - \lambda l}$ and $C_6 = C_2 e^{ikl + \lambda l}$, i.e.

$$\psi_{\mathrm{III}} = e^{ikl}\left(C_1 e^{\lambda(x-l)} + C_2 e^{-\lambda(x-l)}\right)$$

Solutions ψ_{I}, ψ_{II}, ψ_{III} must be continuous with their first derivatives in passing from the first region to the second ($x = 0$) and from the second to the third ($x = a$). This leads us to the following conditions:

$$\psi_{\mathrm{I}}(0) = \psi_{\mathrm{II}}(0) \quad \text{and} \quad C_1 + C_2 = C_3 + C_4$$

$$\left.\frac{d\psi_{\mathrm{I}}}{dx}\right|_{x=0} = \left.\frac{d\psi_{\mathrm{II}}}{dx}\right|_{x=0} \quad \text{and} \quad \lambda(C_1 - C_2) = i\varkappa(C_3 - C_4)$$

$$\psi_{\mathrm{II}}(a) = \psi_{\mathrm{III}}(a) \quad \text{and} \quad C_3 e^{i\varkappa a} + C_4 e^{-i\varkappa a} = e^{ikl}\left(C_1 e^{-\lambda b} + C_2 e^{\lambda b}\right)$$

$$\left.\frac{d\psi_{\mathrm{II}}}{dx}\right|_{x=a} = \left.\frac{d\psi_{\mathrm{III}}}{dx}\right|_{x=a} \quad \text{and} \quad i\varkappa\left(C_3 e^{i\varkappa a} - C_4 e^{-i\varkappa a}\right)$$

$$= \lambda e^{ikl}\left(C_1 e^{-\lambda b} - C_2 e^{\lambda b}\right)$$

Now we have a system of four homogeneous linear equations for the unknowns C_1, C_2, C_3, and C_4. For there to be a nontrivial solution, the system determinant, as we know, must be zero:

$$\begin{vmatrix} 1 & 1 & -1 & -1 \\ \lambda & -\lambda & -i\varkappa & i\varkappa \\ -e^{ikl-\lambda b} & -e^{ikl-\lambda b} & e^{i\varkappa a} & e^{-i\varkappa a} \\ -\lambda e^{ikl-\lambda b} & \lambda e^{ikl-\lambda b} & i\varkappa e^{i\varkappa a} & -i\varkappa e^{-i\varkappa a} \end{vmatrix} = 0$$

This brings us to an equation that determines the energy of the particle in the periodic field:

$$f(E) \equiv \cosh \lambda b \cos \varkappa a + \frac{\lambda^2 - \varkappa^2}{2\lambda x} \sinh \lambda b \sin \varkappa a = \cos kl \qquad (3)$$

On the left the energy enters through $\varkappa$ and l. It is obvious that for this equation to be valid, $|f(E)|$ must not exceed unity. We can see that at $\varkappa a = n\pi$, this is not so, since then $f(E) = \pm \cosh \lambda b$ and $|f(E)| > 1$. Hence, the energies

$$E = \frac{\hbar^2 n^2 \pi^2}{2\mu a^2}$$

prove to be forbidden for a particle in a periodic field (an electron in a crystal).

63. In the limiting case as $V_0 \to \infty$ and $b \to 0$, so that $\lambda b \to 0$ and $\frac{\sinh \lambda_b}{\lambda_b} \to 1$, Eq. (3) of Problem 62 turns into the identity

$$f(E) \equiv \cos \varkappa l + P \frac{\sin \varkappa l}{\varkappa l} = \cos kl \qquad (1)$$

where $P = \lim\limits_{\substack{V_0 \to \infty \\ b \to 0}} \frac{\lambda^2 ab}{2}$. If we denote $\tan \beta = \frac{P}{\varkappa l}$, it takes the form

$$\frac{\cos(\varkappa l - \beta)}{\cos \beta} = \cos kl$$

Obviously, the edges of the allowed energy bands correspond to the conditions $\cos(\varkappa l - \beta) = \pm \cos \beta$, i.e. $\varkappa l = n\pi$ or $\varkappa l - 2\beta = n\pi$. Substituting $\varkappa l = n\pi - \varepsilon$, we get

$$(-1)^n (\cos \varepsilon - \tan \beta \sin \varepsilon) = \cos kl$$

For $0 < \varepsilon \ll l$ the expression on the left is less than one. Consequently, $\varkappa l = n\pi$ are the onsets of the forbidden energy bands. We will find in a similar manner that $\varkappa l = n\pi + 2\beta$ are the onsets of the allowed energy bands. The nth energy band is determined by the $\varkappa l$ that lie in the limits

$$(n-1)\pi + 2\beta \leqslant \varkappa l \leqslant n\pi$$

If $\varkappa l \gg 1$, then $\beta = \arctan \frac{P}{\varkappa l} \approx \frac{P}{n\pi}$ and the width of the forbidden energy band between the nth and the $(n+1)$st allowed energy bands is

$$\Delta(\varkappa_n l) = 2\beta = \frac{2P}{n\pi}$$

If we take the value $\varkappa = \varkappa_0$, for which $kl = 0$, and expand the left-hand and right-hand sides of (1) as $f(E) = A(\varkappa_0) + B(\varkappa - \varkappa_0)$ and $\cos kl = 1 - \frac{(kl)^2}{2}$, from the equation

$$A(\varkappa_0) + B(\varkappa - \varkappa_0) = 1 - \frac{(kl)^2}{2}$$

we get

$$\varkappa = C + Dk^2$$

Since $\varkappa = \sqrt{\frac{2\mu E}{\hbar^2}}$, we have $E = E_0 + k^2F$. The coefficients C, D, E_0 and F can be expressed in terms of A, B and $\varkappa_0$.

64. We introduce $kl = \xi$ and write Eq. (1) of the solution of Problem 63 in the form

$$f(\xi) \equiv \cos\xi + \frac{P\sin\xi}{\xi} = \cos kl$$

In Problem 63 we have found that the right edge of the allowed energy band corresponds to $\xi = n\pi$ and that at this point $f(\xi) = (-1)^n$. In the forbidden energy band, where $\xi > n\pi$, the equation holds for complex values of kl since $|f| > 1$.

If $f(\xi) > 1$, then $\cos kl = \cosh\mu > 1$ and $kl = i\mu$. If $f(\xi) < -1$, then $\cos kl = -\cosh\mu < -1$ and $kl = i\mu + \pi$.

The function $f(\xi) > 1$ for $\xi \geqslant 2n\pi$ and for $\sin\xi > 0$, and $f(\xi) < -1$ when $\xi \geqslant (2n+1)\pi$ and $\sin\xi < 0$. Hence, if we introduce $\varepsilon = \pm 1$ so that $\varepsilon \sin\xi > 0$, then $\cos kl = \varepsilon\cosh\mu$ and in the forbidden energy band the

following condition holds:

$$\varepsilon \cosh \mu = f(\xi) = \frac{P \sin \xi}{\xi} + \cos \xi \tag{1}$$

Next we find the solution for the Schrödinger equation for a semi-infinite crystal. Now condition (1) of Problem 62 holds for none but $n > 0$ (since for $n < 0$ the potential is not periodic) and, hence, $|\rho|$ can also be less than unity. If we introduce $\rho = e^{ikl}$, solutions with complex k become possible provided $\operatorname{Im} k > 0$.

For $x < 0$ the equation

$$-\frac{\hbar^2}{2\mu}\frac{d^2\psi}{dx^2} + W_0\psi = E\psi$$

has the solution

$$\psi_{x<0} = Ae^{\sqrt{q^2-\xi^2}\,\frac{x}{l}}$$

where we denote $q = l\,(2\mu W_0/\hbar^2)^{1/2}$ and $\xi = \varkappa l = l\,(2\mu E/\hbar^2)^{1/2}$. Here $(q^2 - \xi^2)^{1/2} > 0$.

For $x > 0$ (inside the crystal) we consider region I ($0 \leqslant x \leqslant l$), where

$$\psi_{\mathrm{I}} = C_1 e^{i\varkappa x} + C_2 e^{-i\varkappa x} \tag{2}$$

In region II ($l \leqslant x \leqslant 2l$),

$$\psi_{\mathrm{II}} = C_3 e^{i\varkappa x} + C_4 e^{-i\varkappa x} \tag{3}$$

But if $x = 0$, then $x = 0 + l = l$ lies in region I, and we can consider $C_3 = C_1$ and $C_4 = C_2$. By virtue of formula (1) of the solution of Problem 62, we get the condition

$$C_1 e^{i\varkappa l} + C_2 e^{-i\varkappa l} = e^{ikl}(C_1 + C_2)$$

i.e.

$$C_2 = -C_1 \frac{1 - e^{i(\xi - kl)}}{1 - e^{-i(\xi + kl)}} \tag{4}$$

where, as before, $\xi = \varkappa l$. Hence, for $x > 0$, the function is determined by (2) with condition (4).

Because ψ and $\frac{d\psi}{dx}$ must be continuous at $x = 0$,

$$A = C_1 + C_2 \quad \text{and} \quad \frac{A\sqrt{q^2 - \xi^2}}{l} = i\varkappa (C_1 - C_2)$$

whence

$$\sqrt{q^2-\xi^2}=\frac{e^{ikl}-\cos\xi}{\sin\xi}\,\xi \tag{5}$$

Let us consider this equality for complex values of k. Then $e^{ikl} = \varepsilon e^{-\mu}$ (where $\varepsilon \sin \xi \geqslant 0$), and condition (5) gives

$$\sqrt{q^2-\xi^2}\,\frac{\sin\xi}{\xi}+\cos\xi=\varepsilon e^{-\mu} \tag{6}$$

More than that, condition (1) must also hold. If we subtract (6) from (1), we get

$$\left(P-\sqrt{q^2-\xi^2}\right)\frac{\sin\xi}{\xi}=\varepsilon\sinh\mu \tag{7}$$

Since $\sin \xi$ and ε have the same sign and $\sqrt{q^2 - \xi^2}$, ξ, and μ are greater than zero, we get

$$P-\sqrt{q^2-\xi^2}>0$$

i.e.

$$q^2 - P^2 < \xi^2 < q^2$$

Only this condition allows for the existence of additional levels corresponding to complex values of k. We square both sides, subtract (7) from (1), and get an equation that determines the energy levels (ξ):

$$\frac{q^2}{2P}-\sqrt{q^2-\xi^2}=\xi\cot\xi \tag{8}$$

Now let us show that the function corresponding to these energy levels, which lie in the forbidden energy band, decreases as $|x|$ increases on both sides of the boundary "crystal-vacuum" (the plane $x = 0$).

We find that for $x < 0$, the solution $\psi_{x<0} = A \exp\left(\frac{x}{a}\sqrt{q^2 - \xi^2}\right)$ does have this property. For $x > 0$, the solution satisfies the condition $\psi(x + l) = e^{ikl}\psi(x)$ (a periodic field), which we can write in the form

$$\frac{\psi(x)}{e^{ikx}}=\frac{\psi(x+l)}{e^{ik(x+l)}}=u(x)$$

where $u(x)$ is a periodic function. Consequently, for a complex k,

$$\psi(x)=e^{ikx}u(x)=\varepsilon e^{-\mu x/l}u(x)$$

Hence, we have found a state with an energy level lying in the forbidden energy band, and the probability of discovering the particle decreases exponentially on both sides of $x = 0$ (the surface of the crystal). The value ξ, i.e. the position of the energy level, can be found by solving Eq. (8) graphically.

65. Since $\hat{p}_x = -i\hbar\frac{\partial}{\partial x}$ and $\hat{x} = x$, the condition

$$J(\eta) = \int |\hat{p}_x\psi + i\eta\hat{x}\psi|^2 d\tau \geqslant 0$$

yields

$$\langle p_x^2\rangle \langle x^2\rangle \geqslant \frac{\hbar^2}{4} \tag{1}$$

Let us consider the Hamiltonian of the one-dimensional harmonic oscillator:

$$\hat{H} = \frac{\hat{p}^2}{2\mu} + \frac{\mu\omega^2\hat{x}^2}{2}$$

Obviously, $\langle H\rangle = \frac{\langle p^2\rangle}{2\mu} + \frac{\mu\omega^2}{2}\langle x^2\rangle$. Substituting $\frac{\hbar^2}{4\langle x^2\rangle}$ for $\langle p^2\rangle$, we get

$$\langle H\rangle \geqslant \frac{\hbar^2}{8\mu\langle x^2\rangle} + \frac{\mu\omega^2}{2}\langle x^2\rangle = f(\langle x^2\rangle)$$

We choose $\langle x^2\rangle$ that corresponds to the minimum of f, i.e. use the condition

$$\frac{\partial f}{\partial\langle x^2\rangle} = -\frac{\hbar}{8\mu(\langle x_0^2\rangle)^2} + \frac{\mu\omega^2}{2} = 0$$

and substitute it into the expression for $f(\langle x^2\rangle)$:

$$f(\langle x_0^2\rangle) = \frac{\hbar\omega}{2}, \quad \langle H\rangle \geqslant \frac{\hbar\omega}{2}.$$

66. We introduce the coordinate of the centre of mass X and the relative coordinate (separation) $x = x_1 - x_2$, separate the variables, and get

$$\psi = Ce^{i\frac{PX}{\hbar}} e^{-\xi^2/2} H_n(\xi)$$

where $\xi = \sqrt{\frac{\mu\omega}{\hbar}}x$ and $\mu = \frac{m_1 m_2}{m_1 + m_2}$. The energy levels are

$$E = \frac{P^2}{2(m_1+m_2)} + \left(n + \frac{1}{2}\right)\hbar\omega \quad (n = 0, 1, \dots).$$

67. We separate the variables as in the solution to Problem 66. Then we can look for the function in the form

$$\psi = F(R)\,\Phi(r)$$

where $F = e^{i\mathbf{P}\mathbf{R}/\hbar}$ and $\Phi = \Phi_{nlm}(r, \theta, \varphi) = U_{nl}(r) \times \times P_{lm}(\cos\theta) \times e^{im\varphi}$. Here $U_{nl} = e^{-r/(na)} \times \sum\limits_{k=l}^{n-1} a_k \left(\frac{r}{a}\right)^k$ is the same as in formula (III-24), and

$$E_{n,\mathbf{P}} = \frac{\mathbf{P}^2}{2(m+M)} - \frac{\mu e^4}{2\hbar^2 n^2}$$

($\mathbf{P}$ is continuous and $\mu = \frac{mM}{m+M}$, m and M are the masses of the electron and the nucleus, respectively).

68. Obviously, the total potential energy in this problem is

$$V(x_1, x_2) = \frac{k}{2}(x_1^2 + x_2^2) + \frac{k_1}{2}(x_1 - x_2)^2$$

where k and k_1 are the elastic constants that characterize the interaction of the particles with point $x = 0$ and with each other. If we introduce the coordinate of the centre of mass $X = \frac{x_1 + x_2}{2}$ and the relative coordinate (the separation) $x = x_1 - x_2$, we get the equation

$$-\frac{\hbar^2}{2M}\frac{d^2\psi}{dX^2} - \frac{\hbar^2}{2\mu}\frac{d^2\psi}{dx^2} + \frac{M\omega^2}{2}X^2\psi + \mu\frac{\omega_1^2}{2}x^2\psi = E\psi$$

where $M = 2m$ is the total mass of the system, $\mu = m/2$ is the reduced mass, $\omega = \sqrt{\frac{k}{M}} = \sqrt{\frac{k}{2m}}$, and $\omega_1 = \sqrt{\frac{k+2k_1}{2\mu}} = \sqrt{\frac{k+2k_1}{m}}$. We separate the variables and, if we substitute $\psi = f(X)\,F(x)$, we get two one-dimensional equations for harmonic oscillators with natural frequencies ω and ω_1:

$$-\frac{\hbar^2}{2M}\frac{d^2 f}{dX^2} + \frac{M\omega^2}{2}X^2 f = E_1 f$$

$$-\frac{\hbar^2}{2\mu}\frac{d^2 F}{dx^2} + \frac{\mu\omega_1^2}{2}x^2 F = E_2 F$$

By denoting $\xi=\sqrt{\frac{\hbar}{M\omega}}X$ and $u=\sqrt{\frac{\hbar}{\mu\omega_1}}x$, we can by analogy with Problem 46 write the solution as

$$\psi_{n_1n_2}=Ce^{-\xi^2/2}H_{n_1}(\xi)\,e^{-u^2/2}H_{n_2}(u)$$

where H_n is the Hermite polynomial. The energy level that corresponds to this function is

$$E_{n_1n_2}=\left(n_1+\frac{1}{2}\right)\hbar\omega+\left(n_2+\frac{1}{2}\right)\hbar\omega_1.$$

69. In the x-representation the particle in such a potential well with an energy $E_n=\frac{n^2\hbar^2\pi^2}{2ma^2}$ has in the interval $0\leqslant x\leqslant a$ the corresponding function $\psi_n=\sqrt{\frac{2}{a}}\sin\frac{n\pi x}{a}$, for $x>a$ or for $x<0$ the wave function is zero (see the solution to Problem 43).

The momentum distribution is determined by the wave function in the p-representation, to which we can pass from $\psi(x)$ in the usual manner:

$$\varphi(p)=\int_{-\infty}^{\infty}\psi(x)\,\psi_p^*(x)\,dx$$

where $\psi_p(x)=\frac{1}{(2\pi\hbar)^{1/2}}e^{ipx/\hbar}$ are the momentum eigenfunctions in the x-representation, the functions being normalized so that

$$\int_{-\infty}^{\infty}\psi_p^*(x)\,\psi_{p'}(x)\,dx=\delta(p-p')$$

For the given case we substitute $n=2$ and $\psi_2(x)$ and get

$$\varphi(p)=\int_{-\infty}^{\infty}\psi_2(x)\frac{1}{(2\pi\hbar)^{1/2}}e^{ipx/\hbar}\,dx$$

$$=\frac{1}{(\pi\hbar a)^{1/2}}\int_0^a\sin\frac{2\pi x}{a}e^{-ipx/\hbar}\,dx$$

$$=\frac{2}{(\pi\hbar a)^{1/2}}\pi a\hbar^2\frac{e^{-ipa/\hbar}-1}{4\pi^2\hbar^2-p^2a^2}$$

Consequently, the probability of finding the particle with a momentum in the interval from p to $p + dp$ is

$$dW(p) = |\varphi(p)|^2 dp = \frac{32\pi a\hbar^3 \sin^2\left(\frac{pa}{2\hbar}\right)}{(a^2p^2 - 4\pi^2\hbar^2)^2}\, dp.$$

70. In the x-representation $\hat{x} = x$. To pass to a new representation we use the expression for the expectation value of λ:

$$\langle\lambda\rangle = \int \psi^* \hat{L}\psi \, d\tau$$

In a similar expression for $\langle x\rangle$ we substitute $\psi(x)$ expressed in terms of $\varphi(p)$:

$$\psi(x) = \frac{1}{(2\pi\hbar)^{1/2}} \int_{-\infty}^{\infty} \varphi(p)\, e^{ipx/\hbar}\, dp$$

and get

$$\langle x\rangle = \int_{-\infty}^{\infty} \varphi^*(x)\, x \frac{1}{(2\pi\hbar)^{1/2}} \int_{-\infty}^{\infty} \varphi(p)\, e^{ipx/\hbar}\, dp\, dx$$

We change the order of integration over p and x, note that $xe^{ipx/\hbar} = \frac{\hbar}{i}\frac{\partial}{\partial p} e^{ipx/\hbar}$, integrate over p by parts [$\varphi(p) = 0$ at the limits of integration by virtue of the requirement that $|\varphi(p)|^2$ be integrable] and get

$$\langle x\rangle = \int_{-\infty}^{\infty} \psi^*(x) \frac{1}{(2\pi\hbar)^{1/2}} \int_{-\infty}^{\infty} \varphi(p)\, \frac{\hbar}{i} \frac{\partial e^{ipx/\hbar}}{\partial p}\, dp\, dx$$

$$= \int_{-\infty}^{\infty} \frac{\psi^*}{(2\pi\hbar)^{1/2}} \int_{-\infty}^{\infty} e^{ipx/\hbar}\, i\hbar \frac{\partial\varphi}{\partial p}\, dp\, dx$$

Since $\int \psi^* \frac{1}{(2\pi h)^{1/2}} e^{ipx/\hbar}\, dx = \varphi^*(p)$, we have

$$\langle x\rangle = \int_{-\infty}^{\infty} \varphi^*(p)\, i\hbar \frac{\partial}{\partial p} \varphi(p)\, dp$$

i.e.

$$\hat{x}_p = i\hbar \frac{\partial}{\partial p}$$

We look for the eigenfunctions of $\hat{x}_p$ going by the usual rule. These are functions that satisfy both the equation

$$i\hbar \frac{\partial \varphi}{\partial p} = x\varphi(p)$$

and the conditions of finiteness, single-valuedness, and continuity. The solution of this equation

$$\varphi(p) = Ce^{-ixp/\hbar}$$

will satisfy all the conditions for any real value of x; the spectrum of $\hat{x}$ is continuous.

71. For a particle in a homogeneous potential field, $\hat{V} = A\hat{x} = Ai\hbar \frac{\partial}{\partial p}$ (see Problem 70). The equation for the eigenfunctions of the Hamiltonian in the momentum representation has the form

$$\frac{p^2}{2m}\varphi + i\hbar A \frac{\partial \varphi}{\partial p} = E\varphi$$

We separate the variables and write

$$\frac{d\varphi}{\varphi} = \frac{E - \frac{p^2}{2m}}{i\hbar A}\, dp, \quad \text{whence} \quad \varphi - Ce^{\frac{i}{\hbar A}\left(\frac{p^3}{6m} - Ep\right)}$$

The finiteness of $\varphi(p)$ is ensured for any real value of E, i.e. the energy spectrum is continuous. We determine the constant C from the normalization condition

$$\int_{-\infty}^{\infty} \varphi_E^*(p)\, \varphi_{E'}(p)\, dp = \delta(E - E') = C^2 \int_{-\infty}^{\infty} e^{\frac{i}{\hbar A}(E'-E)\, p}\, dp$$

Since $\delta(E - E') = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(E-E')y}\, dy$, we have $C^2 = \frac{1}{2\pi\hbar A}$.

72. We determine the wave functions for the type of potential given in Problem 43.

For $x > \frac{a}{2}$ or $x < -\frac{a}{2}$, the wave function $\psi(x) = 0$.

For $-\frac{a}{2} \leqslant x \leqslant \frac{a}{2}$,

$$\psi(x) = Ae^{ikx} + Be^{-ikx}$$

where $k = (2mE/\hbar^2)^{1/2}$.

The requirement of continuity of the wave function at $x = \pm\frac{a}{2}$ yields

$$\begin{aligned}\psi\left(-\frac{a}{2}\right) &= Ae^{-ika/2} + Be^{ika/2} = 0 \\ \psi\left(\frac{a}{2}\right) &= Ae^{ika/2} + Be^{-ika/2} = 0\end{aligned} \tag{1}$$

whence $ka = n\pi$. Condition (1) reduces to $(A+B)\cos\frac{n\pi}{2} = 0$ or to $(A-B)\sin\frac{n\pi}{2} = 0$, i.e. for n even we have

$$\psi_n(x) = \sqrt{\frac{2}{a}}\sin\frac{n\pi x}{a}$$

and for n odd

$$\psi_n(x) = \sqrt{\frac{2}{a}}\cos\frac{n\pi x}{a}$$

(the functions are normalized in the usual way).

The matrix element $\langle n \mid ex \mid m\rangle = e\int\limits_{-a/2}^{a/2} \psi_n^* x\psi_m\, dx$ is nonzero, obviously, only when n and m have different parities. Otherwise, the integrand will be an odd function and the integral will become zero. Let n be even and m odd. Since the integrand is even, if we introduce $y = \frac{\pi x}{a}$, we can write

$$\begin{aligned}\langle n \mid ex \mid m\rangle &= 2e\frac{2}{a}\int\limits_0^{a/2} x\sin\frac{n\pi x}{a}\cos\frac{m\pi x}{a}\,dx \\ &= \frac{2ae}{\pi^2}\int\limits_0^{\pi/2}[\sin(n+m)y + \sin(n-m)y]\,y\,dy\end{aligned}$$

and since $\int\limits_0^{\pi/2} y\sin(n+m)y\,dy = \frac{\sin(n+m)\frac{\pi}{2}}{(n+m)^2} = \frac{(-1)^{\frac{n+m-1}{2}}}{(n+m)^2}$,

we have

$$\begin{aligned}\langle n \mid ex \mid m\rangle &= \frac{2ae}{\pi^2}\left[\frac{(-1)^{\frac{n+m-1}{2}}}{(n+m)^2} + \frac{(-1)^{\frac{n-m-1}{2}}}{(n-m)^2}\right] \\ &= (-1)^{\frac{n-m+1}{2}}\frac{8aemn}{\pi^2(n^2-m^2)^2}\end{aligned}$$

By analogy the expression $\langle n \mid x^2 \mid m\rangle = \int\limits_{-a/2}^{a/2} x^2 \psi_n^* \psi_m \, dx$ is nonzero only when n and m have the same parities. If n and m are even,

$$\langle n \mid x^2 \mid m\rangle = \frac{4}{a} \int\limits_0^{a/2} x^2 \sin \frac{n\pi x}{a} \sin \frac{m\pi x}{a} \, dx$$

$$= \frac{2a^2}{\pi^3} \int\limits_0^{\pi/2} y^2 [\cos(n-m)y - \cos(n+m)y] \, dy$$

$$= (-1)^{\frac{n-m}{2}} \frac{8a^2 nm}{\pi^2 (n^2-m^2)^2}$$

since

$$\int\limits_0^{\pi/2} y^2 \cos(n-m) y \, dy = \frac{\pi}{(n-m)^2} \cos(n-m) \frac{\pi}{2} = \frac{\pi (-1)^{\frac{n-m}{2}}}{(n-m)^2}$$

For n and m odd

$$\langle n \mid x^2 \mid m\rangle = \frac{4a^2}{\pi^2} \frac{n^2+m^2}{(n^2-m^2)^2}$$

The matrix element of momentum is

$$\langle n \mid p \mid m\rangle = \frac{\hbar}{i} \int\limits_{-a/2}^{+a/2} \psi_n^* \frac{d\psi_m}{dx} \, dx \neq 0$$

if n and m have different parities. For instance, with n even and m odd,

$$\langle n \mid p \mid m\rangle = \frac{\hbar}{i} \frac{2}{a} \int\limits_{-a/2}^{a/2} \sin \frac{n\pi x}{a} \frac{d}{dx} \left(\cos \frac{m\pi x}{a} \right) dx$$

$$= -\frac{\hbar}{i} \frac{2m}{a} \int\limits_0^{\pi/2} [\cos(n-m)y - \cos(n+m)y] \, dy$$

$$= -\frac{\hbar}{i} \frac{2m}{a} \left[\frac{(-1)^{\frac{n-m-1}{2}}}{n-m} - \frac{(-1)^{\frac{n+m-1}{2}}}{n+m} \right]$$

$$= (-1)^{n-m+1} \frac{\hbar}{ia} \frac{4mn}{n^2-m^2}$$

If n and m exchange parity, the result for $\langle n \mid p \mid m \rangle$ will be the same.

73. The equation for the eigenfunctions of a one-dimensional harmonic oscillator in the p-representation has the form

$$\frac{p^2}{2m}\varphi(p)+\frac{m\omega^2}{2}\left(i\hbar\frac{d}{dp}\right)^2\varphi(p)=E\varphi(p)$$

If we introduce the dimensionless variable $\eta = p/(m\hbar\omega)^{1/2}$ and put $\frac{2E}{\hbar\omega}=\lambda$, we arrive at an equation that coincides with the dimensionless equation for the same problem in the x-representation. Referring to Problem 46, we can write

$$\varphi_n(p)=C^{-\eta^2/2}H_n\eta \quad (H_n \text{ is the Hermite polynomial})$$

$$E_n=\left(n+\frac{1}{2}\right)\hbar\omega$$

Hence, for a simple harmonic oscillator the coordinate and momentum distributions are similar.

74. For a simple harmonic oscillator

$$\hat{H}=\frac{\hat{p}^2}{2m}+\frac{m\omega^2\hat{q}^2}{2}$$

and

$$\langle H\rangle=\frac{\langle p^2\rangle}{2m}+\frac{m\omega^2}{2}\langle q^2\rangle\geqslant 0$$

Consequently, $\hat{H}$ has no negative eigenvalues, i.e. $H_n \geqslant 0$.

We introduce

$$\hat{X}=\hat{p}+im\omega\hat{q} \tag{1}$$

$$\hat{X}^{+}=\hat{p}-im\omega\hat{q} \tag{2}$$

and bear in mind that $\hat{p}\hat{q}-\hat{q}\hat{p}=-i\hbar$. Now we can represent $\hat{H}$ either as

$$\hat{H}=\frac{1}{2m}(\hat{p}^2+m^2\omega^2\hat{q}^2)=\frac{1}{2m}[\hat{X}\hat{X}+im\omega(\hat{p}\hat{q}-\hat{q}\hat{p})]$$

$$=\frac{1}{2m}\hat{X}\hat{X}+\frac{\hbar\omega}{2} \tag{3}$$

or as

$$\hat{H}=\frac{1}{2m}\hat{X}+\hat{X}-\frac{\hbar\omega}{2}$$

We construct the commutator

$$\hat{H}\hat{X}-\hat{X}\hat{H}=\left(\frac{1}{2m}\hat{X}\hat{X}^{+}+\frac{\hbar\omega}{2}\right)\hat{X}-\hat{X}\left(\frac{1}{2m}\hat{X}^{+}\hat{X}-\frac{\hbar\omega}{2}\right)=\hbar\omega\hat{X}$$

In like manner

$$\hat{H}\hat{X}^{+}-\hat{X}^{+}\hat{H}=-\hbar\omega\hat{X}^{+}$$

When we have written these commutators in the representation that diagonalizes $\hat{H}$, i.e. $\langle n \mid H \mid n' \rangle = H_n\delta_{nn'}$, we get

$$\hbar\omega\langle n \mid X \mid n'\rangle=\langle n \mid HX \mid n'\rangle-\langle n \mid XH \mid n'\rangle$$

$$=\sum_{n''}[\langle n \mid H \mid n''\rangle\langle n'' \mid X \mid n'\rangle-\langle n \mid X \mid n''\rangle\langle n'' \mid H \mid n'\rangle]$$

$$=(H_n-H_{n'})\langle n \mid X \mid n'\rangle$$

It follows from this that

$$\langle n \mid X \mid n' \rangle (H_n - H_{n'} - \hbar\omega) = 0$$

i.e. $\langle n \mid X \mid n' \rangle$ is nonzero only if $H_{n'} = H_n - \hbar\omega$. By analogy $\langle n \mid X^{+} \mid n' \rangle \neq 0$ if $H_{n'} = H_n + \hbar\omega$. We write the relationship

$$\hat{H}=\frac{1}{2m}\hat{X}^{+}\hat{X}-\frac{\hbar\omega}{2}$$

in the same representation, and for the diagonal matrix element we get

$$\langle n \mid H \mid n\rangle=H_n=\frac{1}{2m}\sum_{n'}\langle n \mid X^{+} \mid n'\rangle\langle n' \mid X \mid n\rangle-\frac{\hbar\omega}{2}$$

In the sum the member with $\langle n \mid X^{+} \mid n' \rangle$ can differ from zero only if $H_{n'} = H_n + \hbar\omega$ is also an eigenvalue of $\hat{H}$. Otherwise, the sum will turn zero and H_n will be equal to $-\frac{\hbar\omega}{2}$, which is impossible in view of what has been said. Consequently, the first assumption must be true, i.e. if H_n is an eigenvalue of $\hat{H}$, then $H_{n'} = H_n + \hbar\omega$ is also its eigenvalue.

Now we consider expression (3). In a similar way we find that

$$H_n=\frac{1}{2m}\sum_{n'}\langle n \mid X \mid n'\rangle\langle n' \mid X^{+} \mid n\rangle+\frac{\hbar\omega}{2} \tag{4}$$

where $H_{n'} = H_n - \hbar\omega$, and either $H_{n'}$ is also an eigenvalue of $\hat{H}$ (i.e. H_n is not the lowest energy level possible) or, if this is not so, $\langle n \mid X \mid n' \rangle = 0$ and $H_n = \frac{\hbar\omega}{2}$ is the lowest energy level, and the next level lies $\hbar\omega$ higher. Hence, we have found that

$$H_n = \left(n + \frac{1}{2}\right) \hbar\omega \tag{5}$$

and $\langle n \mid X \mid n-1 \rangle$ and $\langle n \mid X^+ \mid n+1 \rangle$ differ from zero. If we bear in mind that $(\langle n \mid X \mid n-1 \rangle)^* = \langle n-1 \mid X^+ \mid n \rangle$, we can obtain from (4) and (5) the equation

$$|\langle n \mid X \mid n-1 \rangle|^2 = 2mn\hbar\omega$$

Whence

$$\langle n \mid X \mid n-1 \rangle = (\langle n \mid X^+ \mid n-1 \rangle)^* = \sqrt{2m\hbar\omega n}\, e^{i\varphi}$$

Returning to (1) and (2), we get (choosing $e^{i\varphi} = i$)

$$\langle n \mid q \mid n-1 \rangle = \langle n-1 \mid q \mid n \rangle = \sqrt{\frac{\hbar n}{2m\omega}}$$

$$\langle n \mid p \mid n-1 \rangle = (\langle n-1 \mid p \mid n \rangle)^* = i\sqrt{\frac{m\hbar\omega}{2} n}.$$

75. Let us use the commutation relations for angular momentum

$$\hat{L}_x\hat{L}_y - \hat{L}_y\hat{L}_x = i\hbar\hat{L}_z \quad \text{and} \quad \hat{L}_x\mathbf{L}^2 - \hat{\mathbf{L}}^2\hat{L}_x = 0$$

where $\hat{\mathbf{L}}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2$. Now we introduce

$$\hat{X} = \hat{L}_x + i\hat{L}_y \quad \text{and} \quad \hat{X}^+ = \hat{L}_x - i\hat{L}_y$$

Obviously, $\hat{L}_z$, $\hat{X}$, and $\hat{X}^+$ commute with $\hat{\mathbf{L}}^2$. We write the relationship $\hat{\mathbf{L}}^2\hat{X} - \hat{X}\hat{\mathbf{L}}^2 = 0$ in the representation where $\hat{\mathbf{L}}^2$ and $\hat{L}_z$ are diagonal, i.e. $\langle l \mid \mathbf{L}^2 \mid l' \rangle = L_l^2 \delta_{ll'}$, and we get

$$\langle lm \mid X \mid l'm' \rangle (L_l^2 - L_{l'}^2) = 0$$

Here the only matrix elements that differ from zero are the diagonal in l matrix elements of $\hat{X}$, and also of $\hat{X}^+$ and $\hat{L}_z$. For this reason in future relationships between these operators we can simply consider $\hat{\mathbf{L}}^2$ to be its eigenvalue L_l^2.

Let us represent $\hat{\mathbf{L}}^2$ in the form

$$\hat{\mathbf{L}}^2 = (\hat{L}_x + i\hat{L}_y)(\hat{L}_x - i\hat{L}_y) + i(\hat{L}_x\hat{L}_y - \hat{L}_y\hat{L}_x) + \hat{L}_z^2$$
$$= \hat{X}\hat{X}^+ + \left(\hat{L}_z - \frac{\hbar}{2}\right)^2 - \frac{\hbar^2}{4} \qquad (1)$$

On the other hand,

$$\hat{\mathbf{L}}^2 = \hat{X}^+\hat{X} + \left(\hat{L}_z + \frac{\hbar}{2}\right)^2 - \frac{\hbar^2}{4} \qquad (2)$$

If we construct the commutators of $\hat{L}_z$ with $\hat{X}$ and $\hat{X}^+$, we will see that

$$\hat{L}_z\hat{X} - \hat{X}\hat{L}_z = \hat{L}_z(\hat{L}_x + i\hat{L}_y) - (\hat{L}_x + i\hat{L}_y)\hat{L}_z$$
$$= i\hbar\hat{L}_y + i(-i\hbar\hat{L}_x) = \hbar\hat{X}$$

and, similarly,

$$\hat{L}_z\hat{X}^+ - \hat{X}^+\hat{L}_z = -\hbar\hat{X}^+$$

We write these relationships in the $(\mathbf{L}^2, L_z)$-representation and denote $\langle lm \mid L_z \mid lm' \rangle = m\hbar\delta_{mm'}$ to get

$$\hbar\langle lm \mid X \mid lm'\rangle = \sum_{m'} [\langle lm \mid L_z \mid lm''\rangle\langle lm'' \mid X \mid lm'\rangle$$
$$- \langle lm \mid X \mid lm''\rangle\langle lm'' \mid L_z \mid lm'\rangle] = \langle lm \mid X \mid lm'\rangle(m - m')\hbar$$

i.e.

$$\langle lm \mid X \mid lm' \rangle (m - m' - 1)\hbar = 0$$

Hence, $\langle lm \mid X \mid lm' \rangle \neq 0$ only if $m' = m - 1$. In the same way $\langle lm \mid X^+ \mid lm' \rangle \neq 0$ only if $m' = m + 1$.

Next we write the diagonal (mmth) matrix element of (1) and get

$$L_l^2 = \sum_{m'} \langle lm \mid X \mid lm'\rangle\langle lm' \mid X^+ \mid lm\rangle + \left(m - \frac{1}{2}\right)^2 \hbar^2 - \frac{\hbar^2}{4}$$

In the sum the member that can differ from zero is the one for which $m' = m - 1$. If such an $m'\hbar$ is not an eigenvalue of $\hat{L}_z$, the whole sum is zero and

$$L_l^2 = \left(m - \frac{1}{2}\right)^2 \hbar^2 - \frac{\hbar^2}{4}$$

Otherwise,

$$L_l^2 > \left(m - \frac{1}{2}\right)^2 \hbar^2 - \frac{\hbar^2}{4}$$

Thus, for each L_l^2 there is a certain minimum eigenvalue of $\hat{L}_z$ equal to $m_0\hbar$, so that

$$L_l^2 = \left(m_0 - \frac{1}{2}\right)^2 \hbar^2 - \frac{\hbar^2}{4}$$

and any other eigenvalue of $\hat{L}_z$ must differ from $m_0\hbar$ by an integral multiple of $\hbar$.

Now we write the diagonal element of (2):

$$L_l^2 = \sum_{m'} \langle lm \mid X^+ \mid lm' \rangle \langle lm' \mid X \mid lm \rangle + \left(m + \frac{1}{2}\right)^2 \hbar^2 - \frac{\hbar^2}{4}$$

Here $\langle lm \mid X^+ \mid lm' \rangle \neq 0$ only if there is an eigenvalue of $\hat{L}_z$ equal to $m'\hbar = (m+1)\hbar$. Otherwise, i.e. if $m\hbar = m_1\hbar$ is the greatest value of $\hat{L}_z$ for the given value of L_l^2, we have

$$L_l^2 = \left(m_1 + \frac{1}{2}\right)^2 \hbar^2 - \frac{\hbar^2}{4}$$

It is evident that

$$m_1\hbar = -\frac{\hbar}{2} + \sqrt{L_l^2 + \frac{\hbar^2}{4}} \tag{3}$$

is the greatest eigenvalue of $\hat{L}_z$, and

$$m_0\hbar = \frac{\hbar}{2} - \sqrt{L_l^2 + \frac{\hbar^2}{4}} \tag{4}$$

is the smallest.

The difference $(m_1 - m_0)\hbar = 2\sqrt{L_l^2 + \frac{\hbar^2}{4}} - \hbar$ must be equal, obviously, to $2l\hbar$, where $2l$ is an integer (since all eigenvalues of $\hat{L}_z$ must be separated from each other by an interval of $\hbar$). Hence,

$$2\sqrt{L_l^2 + \frac{\hbar^2}{4}} = (2l+1)\hbar \quad \text{and} \quad L_l^2 = l(l+1)\hbar^2$$

where l can be an integer (if $2l$ is even) or a half-integer. In either case, as we see from the introduction of $2l$, it is nonnegative, i.e. $l = 0, 1, 2, \ldots$ or $l = 1/2, 3/2, \ldots$. Substituting $\sqrt{L_l^2 + \frac{\hbar^2}{4}}$ in (3) and (4), we get

$$m_1 = m_{\max} = l \text{ and } m_0 = m_{\min} = -l$$

Since $\langle lm' \mid X^+ \mid lm\rangle = (\langle lm \mid X \mid lm'\rangle)^*$, we have

$$L_l^2 = |\langle lm | X | l, m-1\rangle|^2 + \left(m - \frac{1}{2}\right)^2 \hbar^2 - \frac{\hbar^2}{4}$$

Hence,

$$(\langle l, m-1 | X^+ | lm\rangle)^* = \langle lm | X | l, m-1\rangle$$

$$= \hbar \sqrt{l(l+1) + \frac{1}{4} - \left(m - \frac{1}{2}\right)^2} = \hbar \sqrt{(l+m)(l-m+1)}$$

It follows from the definition of $\hat{X}$ and $\hat{X}^+$ that

$$\hat{L}_x = \frac{\hat{X} + \hat{X}^+}{2}, \quad \hat{L}_y = \frac{\hat{X} - \hat{X}^+}{2i}$$

and, consequently,

$$\langle lm | L_x | l, m-1\rangle = \frac{\hbar}{2}\sqrt{(l-m+1)(l+m)} = \langle l, m-1 | L_x | lm\rangle$$

$$\langle lm | L_y | l, m-1\rangle = \frac{-i\hbar}{2}\sqrt{(l-m+1)(l+m)}$$

$$= (\langle l, m-1 | L_y | lm\rangle)^*$$

where $m = -l, -l+1, \ldots, l-1, l$.

76. To find the energy distribution, $\langle E\rangle$, and $\langle \Delta E^2\rangle$ we normalize the function $\psi = Ax(a-x)$ and expand it in a complete set of normalized eigenfunctions of the Hamiltonian. These functions are the same as in Problem 72, i.e.

$$\psi_n(x) = \sqrt{\frac{2}{a}} \sin\frac{n\pi x}{a} \quad (n = 1, 2, \ldots)$$

The condition $1 = A^2 \int_0^a x^2 (a-x)^2\, dx$ yields $A^2 = \frac{30}{a^5}$. Now we determine c_n:

$$c_n = \int_0^a \psi(x)\psi_n^*(x)\, dx = A\sqrt{\frac{2}{a}} \int_0^a x(a-x)\sin\frac{n\pi x}{a}\, dx$$

$$= A\sqrt{\frac{2}{a}}\left\{a\left[-\frac{a^2}{n\pi}(-1)^n\right] + \frac{a^3}{n\pi}(-1)^n - \frac{2a^3}{(n\pi)^3}[(-1)^n - 1]\right\}$$

$$= A\sqrt{\frac{2}{a}}\,\frac{2a^3}{(n\pi)^3}[1 - (-1)^n]$$

and we get

$$W(E_n) = |c_n|^2 = \frac{240}{(n\pi)^6}[1-(-1)^n]^2 \neq 0$$

for $n = 1, 3, 5, \ldots$. For even n, on the other hand, the functions ψ and ψ_n have different parity in relation to $x - \frac{a}{2}$. For $n = 1$ $\left(E_1 = \frac{\hbar^2\pi^2}{2\mu a^2}\right)$ we have

$$W(E_1) = \frac{240 \times 2^2}{\pi^6} \approx 0.999$$

i.e. the particle in this state with an overwhelming probability will have an energy equal to E_1.

We can compute the expectation value of the energy as $\langle E\rangle = \sum\limits_n E_n W(E_n)$ or as

$$\langle E\rangle = \int\limits_{-\infty}^{\infty} \psi^* \hat{H}\psi\, dx = A^2 \int\limits_0^a x(a-x)\left(-\frac{\hbar^2}{2\mu}\frac{d^2}{dx^2}\right)$$

$$\times [x(a-x)]\, dx = \frac{5\hbar^2}{\mu a^2}$$

($V = \infty$ for $x < 0$ or $x > a$, but $V\psi = 0$).

We will find that $\langle E\rangle = \sum\limits_{n=1,3,5}^{\infty} \frac{960 n^2}{n^6\pi^6}\frac{\hbar^2\pi^2}{2\mu a^2}$ gives the same result $\langle E\rangle = \frac{10}{\pi^2}E_1$.

To find $\langle \Delta E^2\rangle$ we must first find $\langle E^2\rangle$. Since we cannot consider $V^2\psi$ equal to zero, we calculate the value of $\langle E^2\rangle$ as follows:

$$\langle E^2\rangle = \int\limits_{-\infty}^{\infty} \psi^* \hat{H}^2\psi\, dx = \int\limits_{-\infty}^{\infty} (\hat{H}\psi)^* (\hat{H}\psi)\, dx$$

$$= \int\limits_0^a \left\{-\frac{\hbar^2}{2\mu}\frac{d^2}{dx^2}[Ax(a-x)]\right\}^2 dx = \left(\frac{\hbar^2}{\mu}A\right)^2 a = \frac{30\hbar^4}{\mu^2 a^4}$$

or

$$\langle E^2\rangle = \sum_{n=1,3,5}^{\infty} \frac{960}{n^6\pi^6}\frac{\hbar^4\pi^4 n^4}{4\mu^2 a^4} = \frac{240\hbar^4}{\mu^2 a^4 \pi^2}\sum_{n=1,3,5}^{\infty}\frac{1}{n^2}$$

Whence

$$\langle \Delta E^2 \rangle = \langle E^2 \rangle - (\langle E \rangle)^2 = \frac{5\hbar^4}{\mu^2 a^4}.$$

77. We write the given normalized function in the form

$$\psi = \frac{1}{\sqrt{3\pi}} \left[1 - \frac{1}{2} e^{i2\varphi} - \frac{1}{2} e^{-i2\varphi} \right]$$

It is clear from this that in measuring the angular momentum we can find the values $L_z = 0, +2\hbar, -2\hbar$ with probabilities

$$W(0) = \frac{2}{3}, \quad W(2) = W(-2) = \frac{1}{6}$$

This gives

$$\langle L_z \rangle = 0, \quad \langle L_z^2 \rangle = \frac{4}{3} \hbar^2.$$

78. For the particle localized at point x_0 we can write the wave function as an eigenfunction of $\hat{x}$. In the x-representation, as we know, $\hat{x} = x$, and the equation reduces to $x\psi = x_0\psi$, i.e. $(x - x_0)\psi = 0$ and $\psi(x) \neq 0$ for $x = x_0$. Thus, $\psi(x) = A\delta(x - x_0)$ and the energy spectrum is continuous. The normalization condition

$$A^2 \int_{-\infty}^{\infty} \delta(x - x_0)\, \delta(x - x_0')\, dx = \delta(x_0 - x_0')$$

yields $A = 1$, i.e. $\psi_{x_0}(x) = \delta(x - x_0)$.

In the p-representation, as we know, $\hat{x} = i\hbar \frac{\partial}{\partial p}$ (see the solution to Problem 70), and the eigenfunctions can be obtained from the equation

$$i\hbar \frac{\partial \varphi}{\partial p} = x_0 \varphi$$

from which

$$\varphi_{x_0}(p) = Ce^{-ix_0 p/h}$$

The constant C can be determined from the condition that

$$\int_{-\infty}^{\infty} \varphi_{x_0}(p)\, \varphi_{x_1}^{*}(p)\, dp = \delta(x_1 - x_0)$$

whence

$$C^2 = \frac{1}{2\pi\hbar}$$

since $\delta(x_1 - x_0) = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} e^{i(x_1 - x_0)y}\, dy.$

Similarly, the momentum eigenfunctions are

$$\psi_p(x) = Ce^{ip_0x/\hbar} \quad \text{and} \quad \varphi_p(p) = \delta(p - p_0).$$

79. To write a relationship similar to

$$\psi(\mathbf{r}) = \frac{1}{r}\psi_1(\mathbf{r})$$

in the p-representation, we must transform both functions to this representation. Let

$$\varphi(\mathbf{p}) = \frac{1}{(2\pi\hbar)^{3/2}} \int \psi(\mathbf{r})\, e^{-i\mathbf{pr}/\hbar}\, d\tau$$

and

$$\varphi_1(\mathbf{p}) = \frac{1}{(2\pi\hbar)^{3/2}} \int \psi_1(\mathbf{r})\, e^{-i\mathbf{pr}/\hbar}\, d\tau$$

We substitute $\psi_1(\mathbf{r})/r$ for $\psi(\mathbf{r})$ in the first integral and write $1/r$ in the form of a plane-wave expansion. We assume that

$$\frac{1}{r} = \int a(\mathbf{k})\, e^{i(\mathbf{kr})}\, d\mathbf{k}$$

To find $a(\mathbf{k})$ we apply the Laplace operator to this equality:

$$\Delta \frac{1}{r} = \int a(\mathbf{k})\, \Delta e^{i(\mathbf{kr})}\, d\mathbf{k} = -\int a(\mathbf{k})\, \mathbf{k}^2 e^{i(\mathbf{kr})}\, d\mathbf{k}$$

On the other hand,

$$\Delta \frac{1}{r} = -4\pi\delta(\mathbf{r}) = -\frac{4\pi}{(2\pi)^3} \int e^{i(\mathbf{kr})}\, d\mathbf{k}$$

Hence, $a(\mathbf{k}) = \frac{1}{2\pi^2 k^2}$.

We substitute $\frac{\mathbf{p}_1}{\hbar}$ for $\mathbf{k}$, which yields $d\mathbf{k} = \frac{d\mathbf{p}_1}{\hbar^3}$, and get the final result

$$\frac{1}{r} = \frac{1}{2\pi^2\hbar} \int \frac{1}{p_1^2} e^{i\frac{\mathbf{p}_1\mathbf{r}}{\hbar}}\, d\mathbf{p}_1$$

Then

$$\varphi(\mathbf{p})=\frac{1}{(2\pi\hbar)^{3/2}}\int\frac{1}{r}\psi_1(\mathbf{r})\,e^{-\frac{i\mathbf{pr}}{\hbar}}\,d\tau$$

$$=\frac{1}{2\pi^2\hbar}\int\frac{d\mathbf{p}_1}{p_1^2}\left(\frac{1}{(2\pi\hbar)^{3/2}}\int\psi_1(\mathbf{r})\,e^{-\frac{i(\mathbf{p}-\mathbf{p}_1)\mathbf{r}}{\hbar}}\,d\tau\right)$$

and since the expression in the parentheses is, by definition, $\varphi_1(\mathbf{p}-\mathbf{p}_1)$, we get

$$\psi(\mathbf{p})=\frac{1}{2\pi^2\hbar}\int\frac{\varphi_1(\mathbf{p}-\mathbf{p}_1)}{p_1^2}\,d\mathbf{p}_1$$

Or, after substituting $\mathbf{p}'$ for $\mathbf{p}-\mathbf{p}_1$,

$$\varphi(\mathbf{p})=\frac{1}{2\pi^2\hbar}\int\frac{\varphi_1(\mathbf{p}')\,d\mathbf{p}'}{(\mathbf{p}-\mathbf{p}')^2}$$

The operator $\frac{\hat{1}}{r}$ in the p-representation is an integral operator.

80. The wave function of a particle in a spherically symmetric field has the form

$$\psi_{nlm}=R_{nl}(r)\,P_{lm}(\cos\theta)\,e^{im\varphi}$$

We calculate the matrix element of $D_z=ez$:

$$\langle nlm\,|\,D_z\,|\,n'l'm'\rangle=\int\psi^*_{nlm}ez\psi_{n'l'm'}\,d\tau$$

Evidently, we can represent it in the form of a product of integrals over r, θ, and φ:

$$\langle nlm\,|\,D_z\,|\,n'l'm'\rangle=I_rI_\theta I_\varphi$$

Since the angular part of the wave function is the same for any spherically symmetric field, we can calculate I_θ and I_φ without specifying the potential $V(r)$. Since

$$z=r\cos\theta,\quad d\tau=r^2\sin\theta\,dr\,d\theta\,d\varphi$$

if we assign to I_φ all the multipliers depending on φ, we get

$$I_\varphi=\int_0^{2\pi}e^{-im\varphi}e^{im'\varphi}\,d\varphi=2\pi\delta_{mm'}$$

Consequently,

$$\langle nlm\,|\,D_z\,|\,n'l'm'\rangle=0\ \text{for}\ m'\neq m$$

At $m = m'$, for I_θ we have

$$I_\theta = \int_0^\pi P_{lm}(\cos\theta)\cos\theta\, P_{l'm}(\cos\theta)\sin\theta\, d\theta$$

Introducing $x = \cos\theta$, we write

$$I_\theta = \int_{-1}^{1} P_{lm}(x)\, xP_{l'm}(x)\, dx$$

The Legendre polynomials satisfy the relationship

$$xP_{lm} = aP_{l+1,\,m} + bP_{l-1,\,m}$$

where

$$a = \sqrt{\frac{(l+1)^2 - m^2}{4(l+1)^2 - 1}}, \quad b = \sqrt{\frac{l^2 - m^2}{4l^2 - 1}}, \quad \text{and} \int_{-1}^{1} P_{lm}P_{l'm}\, dx = \delta_{ll'}$$

In view of this

$$I_\theta = \int_{-1}^{1} (aP_{l+1,\,m} + bP_{l-1,\,m})\, P_{l'm}\, dx = a\delta_{l+1,\,l'} + b\delta_{l',\,l-1}$$

and differs from zero only for $l' = l \pm 1$.

To calculate the matrix elements of D_x and D_y it is convenient to introduce the notation

$$D_\pm = D_x \pm iD_y = er\sin\theta\, e^{\pm i\varphi}$$

Once we have

$$\langle nlm \,|\, D_\pm \,|\, n'l'm'\rangle = \int \psi^*_{nlm}\, r\sin\theta e^{\pm i\varphi}\psi_{n'l'm'}\, d\tau = I'_r I'_\theta I'_\varphi$$

we can calculate I'_φ and I'_θ. Since

$$I'_\varphi = \int_0^{2\pi} e^{-im\varphi}\, e^{\pm i\varphi} e^{im'\varphi}\, d\varphi = 2\pi\delta_{m',\,m\mp 1}$$

the matrix element of D_+ differs from zero only at $m' = m - 1$, and that of D_- at $m' = m + 1$. We calculate I'_θ for D_+ assuming that $m' = m - 1$:

$$I'_\theta = \int_0^\pi P_{lm}(\cos\theta)\sin\theta\, P_{l',\,m-1}(\cos\theta)\sin\theta\, d\theta$$

Substituting $x = \cos\theta$ $(\sin\theta = \sqrt{1-x^2})$, we get

$$I'_\theta = \int_{-1}^{1} P_{lm}(x)\sqrt{1-x^2}\,P_{l',\,m-1}(x)\,dx$$

But the theory of the Legendre polynomials yields

$$\sqrt{1-x^2}\,P_{l',\,m-1} = a_1 P_{l'+1,\,m} + b_1 P_{l'-1,\,m}$$

where

$$a_1 = \sqrt{\frac{(l'+m)\,(l'+m+1)}{4\,(l'+1)^2-1}}, \quad b_1 = -\sqrt{\frac{(l'-m+1)\,(l'-m)}{4\,(l')^2-1}}$$

and, consequently,

$$I'_\theta = a_1\delta_{l,\,l'+1} + b_1\delta_{l,\,l'-1}$$

which is not equal to zero at $l' = l \pm 1$.

For D_-, if we assume that $m' = m+1$, we get

$$I'_\theta = \int_{-1}^{1} P_{lm}(x)\sqrt{1-x^2}\,P_{l',\,m+1}(x)\,dx$$

Substituting $\sqrt{1-x^2}\,P_{lm} = a_2 P_{l+1,\,m+1} + b_2 P_{l-1,\,m+1}$, where

$$a_2 = \sqrt{\frac{(l+m+1)\,(l+m+2)}{4\,(l+1)^2-1}}, \quad b_2 = -\sqrt{\frac{(l-m)\,(l-m-1)}{4l^2-1}}$$

we get

$$I'_\theta = a_2\delta_{l+1,\,l'} + b_2\delta_{l-1,\,l'}$$

The expressions for the matrix elements of D_x and D_y can be obtained in an obvious way:

$$\langle nlm\,|\,D_x\,|\,n'l'm'\rangle = \frac{1}{2}\,[\langle nlm\,|\,D_+\,|\,n'l'm'\rangle + \langle nlm\,|\,D_-\,|\,n'l'm'\rangle]$$

$$\langle nlm\,|\,D_y\,|\,n'l'm'\rangle = \frac{1}{2i}\,[\langle nlm\,|\,D_+\,|\,n'l'm'\rangle - \langle nlm\,|\,D_-\,|\,n'l'm'\rangle]$$

It is evident that both differ from zero for $l' = l \pm 1$ and $m' = m \pm 1$.

82. We use the general definition

$$\frac{d\hat{L}}{dt} = \frac{\partial\hat{L}}{\partial t} + \frac{i}{\hbar}(\hat{H}\hat{L} - \hat{L}\hat{H})$$

Since $\hat{H}=\frac{\hat{\mathbf{p}}^2}{2\mu}+\hat{V}(\mathbf{r})$, $\frac{\partial \mathbf{r}}{\partial t}=0$, and $\hat{x}\hat{y}-\hat{y}\hat{x}=0$ for any two components of $\mathbf{r}$, we find that $\hat{\mathbf{r}}\hat{V}-\hat{V}\hat{\mathbf{r}}=0$ and

$$\frac{d\hat{\mathbf{r}}}{dt}=\frac{i}{2\mu\hbar}(\hat{\mathbf{p}}^2\hat{\mathbf{r}}-\hat{\mathbf{r}}\hat{\mathbf{p}}^2)$$

We write $\hat{\mathbf{r}}=\mathbf{i}\hat{x}+\mathbf{j}\hat{y}+\mathbf{k}\hat{z}$ and $\hat{\mathbf{p}}^2=\hat{p}_x^2+\hat{p}_y^2+\hat{p}_z^2$ and we use the basic commutation relations of quantum mechanics, $\hat{p}_x\hat{y}-\hat{y}\hat{p}_x=i\hbar\delta_{xy}$, etc. Thus we get

$$\frac{d\hat{\mathbf{r}}}{dt}=\frac{i}{2\mu\hbar}[\mathbf{i}(\hat{p}_x^2\hat{x}-\hat{x}\hat{p}_x^2)+\mathbf{j}(\hat{p}_y^2\hat{y}-\hat{y}\hat{p}_y^2)+\mathbf{k}(\hat{p}_z^2\hat{z}-\hat{z}\hat{p}_z^2)]$$

But $\hat{p}_x^2\hat{x}-\hat{x}\hat{p}_x^2=\hat{p}_x^2\hat{x}-p_x\hat{x}p_x+\hat{p}_x\hat{x}\hat{p}_x-\hat{x}\hat{p}_x^2$. Now if we group the members in pairs, we can factor out the common multiplier $\hat{p}_x$ from the first pair to the left and from the second pair to the right. Then $\hat{p}_x^2\hat{x}-\hat{x}\hat{p}_x^2=-2i\hbar p_x$ and, hence,

$$\frac{d\hat{\mathbf{r}}}{dt}=\frac{i}{2\mu\hbar}(-2i\hbar)\{\mathbf{i}\hat{p}_x+\mathbf{j}\hat{p}_y+\mathbf{k}\hat{p}_z\}=\frac{\hat{\mathbf{p}}}{\mu}$$

The operator $\hat{\mathbf{p}}$ does not explicitly depend on time either, and

$$\frac{d\hat{\mathbf{p}}}{dt}=\frac{i}{\hbar}[\hat{H}\hat{\mathbf{p}}-\hat{\mathbf{p}}\hat{H}]$$

Since $\hat{p}_x\hat{p}_y-\hat{p}_y\hat{p}_x=0$, the expression reduces to

$$\frac{d\hat{\mathbf{p}}}{dt}=\frac{i}{\hbar}(\hat{V}\hat{\mathbf{p}}-\hat{\mathbf{p}}\hat{V})$$

We now compute it in the x-representation. Then $\hat{\mathbf{p}}=-i\hbar\nabla$ and $V=V(x, y, z)$. Computing

$$(V\mathbf{p}-\mathbf{p}V)\psi(\mathbf{r})=-i\hbar[V\operatorname{grad}\psi-\operatorname{grad}(V\psi)]=i\hbar(\operatorname{grad}V)\psi$$

we get

$$\frac{d\hat{\mathbf{p}}}{dt}=-\operatorname{grad}V=\mathbf{F}$$

Since the commutation relations do not change in transforming to other representations, this relationship always holds.

83. Since the operators $\hat{\mathbf{r}}$ and $\hat{\dot{\mathbf{r}}} = -\frac{i\hbar}{m}\nabla$ do not commute (see the solution to Problem 82), the operator $\hat{\mathbf{j}}$ must be represented in a symmetrized form, i.e.

$$\hat{\mathbf{j}} = -\frac{i\hbar e}{2m}[\delta(\mathbf{r}-\mathbf{r}_0)\nabla - \nabla\delta(\mathbf{r}-\mathbf{r}_0)]$$

Following the general rule, we compose $\langle \mathbf{j} \rangle = \int \psi^*(\mathbf{r})\,\hat{\mathbf{j}}\psi(\mathbf{r})d\tau$. Then we integrate the second term by parts and, using the property of the delta function, we get the final expression for $\langle \mathbf{j} \rangle$:

$$\langle \mathbf{j} \rangle = -\frac{i\hbar e}{2m}(\psi^* \operatorname{grad}\psi - \psi \operatorname{grad}\psi^*)_{\mathbf{r}=\mathbf{r}_0}.$$

84. According to the usual rules of commutation,

$$\frac{d\hat{L}_z}{dt} = \hat{x}\frac{\partial \hat{V}}{\partial y} - \hat{y}\frac{\partial \hat{V}}{\partial x}$$

and $\hat{L}_z$ is an integral of motion in a field with an axis of symmetry OZ. In a central force field, for $V = V(r)$, we have

$$\frac{d\hat{\mathbf{L}}^2}{dt} = i\hbar(\Delta_{\theta\varphi}V - V\Delta_{\theta\varphi}) = 0.$$

85. In composing the operators $\hat{K}_x$, $\hat{K}_y$, and $\hat{K}_z$ we must symmetrize the products of type $\hat{v}_y\hat{L}_z$ since the operators $\hat{v}_y$ and $\hat{L}_z$ do not commute. Hence,

$$\hat{K}_x = \frac{1}{2}(\hat{v}_y\hat{L}_z + \hat{L}_z\hat{v}_y - \hat{v}_z\hat{L}_y - \hat{L}_y\hat{v}_z) + \frac{\alpha x}{r}$$

By analogy we determine $\hat{K}_y$ and $\hat{K}_z$.

Since the field is spherically symmetric, $\hat{H}$ will commute with $\hat{L}_x$, $\hat{L}_y$, $\hat{L}_z$. We must keep this in mind when we compose commutators. To prove that $\frac{d\hat{K}_x}{dt} = 0$ without involved computations we must calculate the commutators of $\hat{H}$ with each of the operators $\hat{v}_y$, $\hat{v}_z$, and $\frac{\hat{x}}{r}$ separately. These

are

$$[\hat{H}, \hat{v}_y] = \frac{1}{m}[\hat{H}, \hat{p}_y] = -\frac{i\hbar\alpha}{m}\frac{y}{r^3}$$

$$[\hat{H}, \hat{v}_z] = -\frac{i\hbar\alpha}{m}\frac{z}{r^3}$$

$$\left[\hat{H}, \frac{\hat{x}}{r}\right] = \frac{\hbar^2}{m}\left[\frac{x}{r^3} - \frac{i}{\hbar}\left(\frac{\hat{p}_x}{r} - \frac{x(\mathbf{r}\cdot\hat{\mathbf{p}})}{r^3}\right)\right]$$

If we substitute these expressions into

$$\frac{d\hat{K}_x}{dt} = \frac{i}{\hbar}[\hat{H}, \hat{K}_x]$$

we see that $\frac{d\hat{K}_x}{dt} = 0$.

86. If we use the results of Problems 31 and 32, we can easily find that

$$\frac{d\hat{a}}{dt} = -i\omega\hat{a} \quad \text{and} \quad \frac{d\hat{a}^*}{dt} = i\omega\hat{a}^*.$$

87. We use the commutation relations for coordinates and momenta, $\hat{p}_x\hat{y} - \hat{y}\hat{p}_x = -i\hbar\delta_{xy}$, and first write $\frac{d\hat{x}}{dt}$. For simplicity, let us denote $\hat{P}_x = \hat{p}_x - eA_x$. Since $\hat{x}$ commutes with $\hat{P}_y$ and $\hat{P}_z$ (and, all the more so, with $e\varphi$), we get

$$\frac{d\hat{x}}{dt} = \frac{i}{\hbar}(\hat{H}\hat{x} - \hat{x}\hat{H}) = \frac{i}{2\hbar\mu}(\hat{P}_x^2\hat{x} - \hat{x}\hat{P}_x^2)$$

$$= \frac{i}{2\hbar\mu}[\hat{P}_x(\hat{P}_x\hat{x} - \hat{x}\hat{P}_x) + (\hat{P}_x\hat{x} - \hat{x}\hat{P}_x)\hat{P}_x] = \frac{\hat{P}_x}{\mu}$$

Next we find $\frac{d\hat{P}_x}{dt} = \frac{d}{dt}(\hat{p}_x - eA_x)$. Since $\mathbf{A}(\mathbf{r}, t)$ can depend on t, we have $\frac{\partial\hat{P}_x}{\partial t} = -e\frac{\partial A_x}{\partial t}$ and

$$\frac{d\hat{P}_x}{dt} = -e\frac{\partial A_x}{\partial t} + \frac{i}{2\mu\hbar}[\hat{P}_y^2\hat{P}_x - \hat{P}_x\hat{P}_y^2 + \hat{P}_z^2\hat{P}_x - \hat{P}_x\hat{P}_z^2] + \frac{ie}{\hbar}(\varphi\hat{P}_x - \hat{P}_x\varphi)$$

Obviously,

$$\varphi\hat{P}_x - \hat{P}_x\varphi = \varphi\hat{p}_x - \hat{p}_x\varphi = i\hbar\frac{\partial\varphi}{\partial x}$$

Denote

$$\hat{D} = \hat{P}_y^2\hat{P}_x - \hat{P}_x\hat{P}_y^2 = \hat{P}_y(\hat{P}_y\hat{P}_x - \hat{P}_x\hat{P}_y) + (\hat{P}_y\hat{P}_x - \hat{P}_x\hat{P}_y)\hat{P}_y$$

First we find

$$\hat{P}_y\hat{P}_x - \hat{P}_x\hat{P}_y = (\hat{p}_y - eA_y)(\hat{p}_x - eA_x) - (\hat{p}_x - eA_x)(\hat{p}_y - eA_y)$$

$$= -e(\hat{p}_yA_x - A_x\hat{p}_y + A_y\hat{p}_x - \hat{p}_xA_y)$$

Since $\hat{p}_yA_x - A_x\hat{p}_y = -i\hbar\dfrac{\partial A_x}{\partial y}$, we have

$$\hat{P}_y\hat{P}_x - \hat{P}_x\hat{P}_y = -i\hbar e\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) = -i\hbar e(\text{curl}\,\mathbf{A})_z$$

and

$$\hat{D} = -i\hbar e(\hat{P}_yB_z + B_z\hat{P}_y), \quad \text{where} \quad \mathbf{B} = \text{curl}\,\mathbf{A}$$

Thus,

$$\frac{d\hat{P}_x}{dt} = -e\frac{\partial A_x}{\partial t} - e\frac{\partial \varphi}{\partial x} + \frac{e}{2\mu}[\hat{P}_yB_z + B_z\hat{P}_y - \hat{P}_zB_y - B_y\hat{P}_z]$$

and since $-\dfrac{\partial A_x}{\partial t} - \dfrac{\partial \varphi}{\partial x} = E_x$, we get the Lorentz force equation in the operator form:

$$\frac{d\hat{P}_x}{dt} = -eE_x + \frac{e}{2}(\hat{\dot{y}}B_z + B_z\hat{\dot{y}} - \hat{\dot{z}}B_y - B_y\hat{\dot{z}})$$

where **E** is the electric field vector, and **B** the vector of magnetic induction.

88. The operator $\hat{x}_1$ does not depend explicitly on time and commutes with $\hat{\alpha}_i$, $\hat{\alpha}_4$, **A**, and φ. If we use the commutation relations of coordinates and momenta:

$$\hat{p}_i\hat{x}_k - \hat{x}_k\hat{p}_i = -i\hbar\delta_{ik}$$

we get

$$\frac{d\hat{x}_1}{dt} = \frac{i}{\hbar}[c\hat{\alpha}_1(\hat{p}_1 - eA_1)\hat{x}_1 - \hat{x}_1c\hat{\alpha}_1(\hat{p}_1 - eA_1)]$$

$$= \frac{i}{\hbar}c\hat{\alpha}_1(\hat{p}_1\hat{x}_1 - \hat{x}_1\hat{p}_1) = c\hat{\alpha}_1$$

We denote $\hat{P}_1 = \hat{p}_1 - eA_1$ and calculate $\frac{d\hat{P}_1}{dt}$:

$$\frac{d}{dt}(\hat{p}_1 - eA_1) = -e\frac{\partial A_1}{\partial t} + \frac{i}{\hbar}[c\hat{\alpha}_2(\hat{P}_2\hat{P}_1 - \hat{P}_1\hat{P}_2) + c\hat{\alpha}_3(\hat{P}_3\hat{P}_1 - \hat{P}_1\hat{P}_3)] + \frac{i}{\hbar}e(\varphi\hat{P}_1 - \hat{P}_1\varphi)$$

The operators $\hat{P}_i$ commute with $\hat{\alpha}_i$. Besides (see the solution to Problem 87),

$$\hat{P}_1\hat{P}_2 - \hat{P}_2\hat{P}_1 = i\hbar\frac{e}{c}B_3, \quad \text{where} \quad \mathbf{B} = \text{curl } \mathbf{A}$$

and

$$\varphi\hat{P}_1 - \hat{P}_1\varphi = \varphi\hat{p}_1 - \hat{p}_1\varphi = i\hbar\frac{\partial\varphi}{\partial x_1}$$

Thus,

$$\frac{d\hat{P}_1}{dt} = -e\frac{\partial A_1}{\partial t} - e\frac{\partial\varphi}{\partial x_1} + e[c\hat{\alpha}_2 B_3 - c\hat{\alpha}_3 B_2]$$

Since $c\hat{\alpha}_i = \hat{\dot{x}}_i$, we again get the Lorentz force equation in the operator form.

90. For a system of N particles the operator of the total momentum is $\hat{\mathbf{P}} = \sum_{i=1}^{N} \hat{\mathbf{p}}_i$ and the Hamiltonian is

$$\hat{H} = \sum_{i=1}^{N} \frac{\hat{\mathbf{p}}_i^2}{2m_i} + \sum_{i=1}^{N} U_i(\mathbf{r}_i) + \sum_{i>k}^{N} U_{ik}(\mathbf{r}_{ik})$$

where $\sum_{i=1}^{N} U_i(\mathbf{r}_i)$ is the total potential energy in an external field, and $\sum_{i>k} U_{ik}(\mathbf{r}_{ik})$ is the energy of interaction between the particles of the system.

Applying the commutation relations, we find that

$$\frac{d\hat{\mathbf{P}}}{dt} = -\sum_{k=1}^{N} \text{grad}_k U_k.$$

91. Since $V(x)$ has a discontinuity at $x = 0$, we must denote the region $x < 0$ as region I ($V = 0$) and the region $x > 0$ as region II ($V = V_0$). After solving the Schrödinger equation in these regions, we must fit the solutions,

i.e. at point $x = 0$ equate the functions and equate their first derivatives.

In region I the Schrödinger equation becomes

$$\frac{d^2\psi_I}{dx^2} + k_1^2\psi_I = 0$$

where $k_1^2 = \frac{2mE}{\hbar^2}$, and its solution is

$$\psi_I = C_1 e^{ik_1x} + C_2 e^{-ik_1x} \tag{1}$$

In region II the Schrödinger equation is

$$\frac{d^2\psi_{II}}{dx^2} + k_2^2\psi_{II} = 0$$

where $k_2^2 = \frac{2m(E - V_0)}{\hbar^2}$; $k_2^2 > 0$ for $E > V_0$, and $k_2^2 < 0$ for $E < V_0$. Its solution is

$$\psi_{II.} = C_3 e^{ik_2x} + C_4 e^{-ik_2x} \tag{2}$$

We fit the solutions and their derivatives:

$$\psi_I(0) = \psi_{II}(0), \qquad C_1 + C_2 = C_3 + C_4$$

$$\left.\frac{d\psi_I}{dx}\right|_{x=0} = \left.\frac{d\psi_{II}}{dx}\right|_{x=0}, \qquad ik_1(C_1 - C_2) = ik_2(C_3 - C_4)$$

Four constants must satisfy two equations. Since one of the constants can be chosen arbitrarily and in region II for physical reasons we can expect to find none but a particle moving in the positive direction, i.e. $p_x = \hbar k_2 > 0$, we must put $C_4 = 0$. If $E < V_0$, then $k_2 = i\alpha$ and $e^{-ik_2x} = e^{+\alpha x} \to \infty$ as $x \to \infty$. For ψ_{II} to be finite at every point, C_4 must be zero. The equations give us

$$\frac{C_2}{C_1} = \frac{k_1 - k_2}{k_1 + k_2} \quad \text{and} \quad \frac{C_3}{C_1} = \frac{2k_1}{k_1 + k_2}$$

Now we must determine the reflectance and transmittance as ratios of corresponding current densities:

$$R = \left|\frac{j_{\text{refl}}}{j_{\text{in}}}\right| \quad \text{and} \quad D = \left|\frac{j_{\text{trans}}}{j_{\text{in}}}\right|$$

For $E > V_0$

$$j_{\text{trans}} = \frac{i\hbar}{2m}\left(\psi_{II}\frac{d\psi_{II}^*}{dx} - \psi_{II}^*\frac{d\psi_{II}}{dx}\right) = \frac{\hbar k_2}{m}|C_3|^2$$

In expression (1) the first member represents the particle moving in the positive direction of the x-axis (the particle is falling on the step), and the second represents the reflec-

ted one. With the aid of these functions we compose the corresponding current densities:

$$j_{\text{in}} = \frac{\hbar k_1}{m} |C_1|^2 \quad \text{and} \quad j_{\text{refl}} = -\frac{\hbar k_1}{m} |C_2|^2$$

and we get

$$R = \left| \frac{k_1 - k_2}{k_1 + k_2} \right|^2, \qquad D = \left| \frac{4k_1 k_2}{(k_1 + k_2)^2} \right|$$

For $E < V_0$ we find that $k_2 = i\alpha$ and that the function $\psi_{\text{II}} = C_3 e^{-\alpha x}$ is a real function and diminishes as x moves away from zero. Evidently, in this case $j_{\text{trans}} = 0$ and $D = 0$. Accordingly,

$$R = \left| \frac{k_1 - i\alpha}{k_1 + i\alpha} \right|^2 = 1.$$

92. As in Problem 91, $R = \left| \frac{j_{\text{refl}}}{j_{\text{in}}} \right|$ and $D = \left| \frac{j_{\text{trans}}}{j_{\text{in}}} \right|$. We find the functions that characterize the incoming, reflected, and transmitted particles (waves).

We denote as region I the region $x < 0$ $(V = 0)$ and introduce $k_1^2 = \frac{2mE}{\hbar^2}$. As region II we denote the region $0 \leqslant x \leqslant a$ $(V = V_0)$ and $k_2^2 = \frac{2m(E - V_0)}{\hbar^2}$. As region III we denote $x > a$ $(V = 0)$. Next we write the Schrödinger equation and its solution in each region

$$\frac{d^2\psi_{\text{I}}}{dx^2} + k_1^2 \psi_{\text{I}} = 0, \qquad \psi_{\text{I}} = C_1 e^{ik_1 x} + C_2 e^{-ik_1 x}$$

$$\frac{d^2\psi_{\text{II}}}{dx^2} + k_2^2 \psi_{\text{II}} = 0, \qquad \psi_{\text{II}} = C_3 e^{ik_2 x} + C_4 e^{-ik_2 x}$$

$$\frac{d^2\psi_{\text{III}}}{dx^2} + k_1^2 \psi_{\text{III}} = 0, \qquad \psi_{\text{III}} = C_5 e^{ik_1 x} + C_6 e^{-ik_1 x}$$

We assume that $C_6 = 0$ since in region III there is none but the transmitted wave (see the solution to Problem 91). Requiring that ψ and $\frac{d\psi}{dx}$ be continuous at $x = 0$ and $x = a$, we get four equations for the C's:

$$\psi_{\text{I}}(0) = \psi_{\text{II}}(0), \qquad C_1 + C_2 = C_3 + C_4;$$

$$\left.\frac{d\psi_{\text{I}}}{dx}\right|_{x=0} = \left.\frac{d\psi_{\text{II}}}{dx}\right|_{x=0}, \qquad ik_1(C_1 - C_2) = ik_2(C_3 - C_4);$$

$$\psi_{II}(a)=\psi_{III}(a), \qquad C_3e^{ik_2a}+C_4e^{-ik_2a}=C_5e^{ik_1a};$$

$$\left.\frac{d\psi_{II}}{dx}\right|_{x=a}=\left.\frac{d\psi_{III}}{dx}\right|_{x=a}, \quad ik_2\left(C_3e^{ik_2a}-C_4e^{-ik_2a}\right)=ik_1C_5e^{ik_1a}$$

Now we can determine the ratios of different C_i to C_1 ($i=2, 3, 4, 5$). Bearing in mind that $j=\frac{i\hbar}{2m}\left(\psi\frac{d\psi^*}{dx}-\right.$ $\left.-\psi^*\frac{d\psi}{dx}\right)$, we find that

$$j_{\text{in}}=\frac{\hbar k_1}{m}|C_1|^2, \quad j_{\text{refl}}=-\frac{\hbar k_1}{m}|C_2|^2, \quad j_{\text{trans}}=\frac{\hbar k_1}{m}|C_5|^2$$

and $R=|C_2/C_1|^2$, $D=|C_5/C_1|^2$.

$$R=\left|\frac{C_2}{C_1}\right|^2, \qquad D=\left|\frac{C_5}{C_1}\right|^2$$

From the four equations we determine C_2/C_1 and C_5/C_1:

$$\frac{C_2}{C_1}=\frac{\Delta_2}{\Delta_1}, \qquad \frac{C_5}{C_1}=\frac{\Delta_5}{\Delta_1}$$

where

$$\Delta_1=\begin{vmatrix} 1 & -1 & -1 & 0 \\ -1 & -\frac{k_2}{k_1} & \frac{k_2}{k_1} & 0 \\ 0 & e^{ik_2a} & e^{-ik_2a} & -e^{ik_1a} \\ 0 & e^{ik_2a} & -e^{-ik_2a} & -\frac{k_1}{k_2}e^{ik_1a} \end{vmatrix}$$

$$=4e^{ik_1a}\left[\cos k_2a-\frac{i\left(k_1^2+k_2^2\right)\sin k_2a}{2k_1k_2}\right];$$

$$\Delta_2=\begin{vmatrix} -1 & -1 & -1 & 0 \\ -1 & -\frac{k_2}{k_1} & \frac{k_2}{k_1} & 0 \\ 0 & e^{ik_2a} & e^{-ik_2a} & -e^{ik_1a} \\ 0 & e^{ik_2a} & -e^{-ik_2a} & -\frac{k_1}{k_2}e^{ik_1a} \end{vmatrix}$$

$$=\frac{2e^{ik_1a}}{k_1k_2}\, i\left(k_2^2-k_1^2\right)\sin k_2a;$$

$$\Delta_5 = \begin{vmatrix} 1 & -1 & -1 & -1 \\ -1 & -\frac{k_2}{k_1} & \frac{k_2}{k_1} & -1 \\ 0 & e^{ik_2a} & e^{-ik_2a} & 0 \\ 0 & e^{ik_2a} & -e^{-ik_2a} & 0 \end{vmatrix} = 4$$

Now we substitute the calculated values of Δ_1, Δ_2, Δ_5 into the expression for R and D, and, considering $E > V_0$ (k_2 is real) and simplifying, we get

$$R = \frac{(k_2^2 - k_1^2)^2 \sin^2 k_2a}{|2k_1k_2 \cos k_2a - (k_1^2 + k_2^2)\, i \sin k_2a|^2} = \frac{(k_2^2 - k_1^2)^2 \sin^2 k_2a}{4k_1^2k_2^2 + (k_2^2 - k_1^2)^2 \sin^2 k_2a}$$

$$D = \frac{4k_1^2k_2^2}{4k_1^2k_2^2 + (k_2^2 - k_1^2)^2 \sin^2 k_2a}$$

It is easy to see that $D + R = 1$ and that for $k_2a = n\pi$ the barrier is transparent, i.e. $D = 1$ and $R = 0$. The solution holds for $V_0 > 0$ and for $V_0 < 0$ (when the particle passes over a potential well).

If we consider the case of $V_0 > 0$ and $E < V_0$, we get an imaginary $k_2 = i\beta$. Then $\sin k_2a = i \sinh \beta a$, and we find the expressions for R and D:

$$R = \frac{(k_1^2 + \beta^2)^2 \sinh^2 \beta a}{4k_1^2\beta^2 + (k_1^2 + \beta^2)^2 \sinh^2 \beta a}$$

$$D = \frac{4k_1^2\beta^2}{4k_1^2\beta^2 + (k_1^2 + \beta^2)^2 \sinh^2 \beta a}$$

For $\beta a \gg 1$ we get

$$D \approx \frac{16k_1^2\beta^2}{(k_1^2 + \beta^2)^2} e^{-2\beta a}.$$

93. Let region I be the region $0 \leqslant x \leqslant a$, region II be $a \leqslant x \leqslant b$, and region III be $x \geqslant b$.

We note that $V = \infty$ for $x < 0$. Hence, $\psi = 0$. If we choose ψ_{I} so that continuity is ensured at $x = 0$, we can write the solution of the Schrödinger equation in the re-

gions I, II, III:

$$\frac{d^2\psi_{\mathrm{I}}}{dx^2}+k^2\psi_{\mathrm{I}}=0, \quad \text{where} \quad k^2=\frac{2mE}{\hbar^2} \quad \text{and} \quad \psi_{\mathrm{I}}=A\sin kx;$$

$$\frac{d^2\psi_{\mathrm{II}}}{dx^2}-\varkappa^2\psi_{\mathrm{II}}=0, \quad \text{where} \quad \varkappa^2=\frac{2m}{\hbar^2}(V_0-E) \quad \text{and}$$

$$\psi_{\mathrm{II}}=B_1e^{\varkappa(x-a)}+B_2e^{-\varkappa(x-a)};$$

$$\frac{d^2\psi_{\mathrm{III}}}{dx^2}+k^2\psi_{\mathrm{III}}=0, \quad \psi_{\mathrm{II}}=e^{ik(x-b)}+Ce^{-ik(x-b)}$$

The solution is normalized in such a way that the amplitude of the wave leaving the potential well is unity. This does not restrict its general character since all the equations are homogeneous and determine none but the ratios of the coefficients.

We write the continuity conditions on the boundaries $x = a$ and $x = b$ for the function and its first derivative:

$$A\sin ka=B_1+B_2 \tag{1}$$

$$kA\cos ka=\varkappa(B_1-B_2) \tag{2}$$

$$B_1e^{\varkappa l}+B_2e^{-\varkappa l}=1+C \tag{3}$$

$$\varkappa(B_1e^{\varkappa l}-B_2e^{-\varkappa l})=ik(1-C) \tag{4}$$

Here we have introduced $l = b - a$ as the width of the barrier. If we add and subtract (1) and (2), we find B_1 and B_2. Then if we substitute them into conditions (3) and (4) and repeat the operations of adding and subtracting, we get equations that define the amplitude A of the wave in the inner region and the amplitude C of the wave falling on the barrier:

$$\frac{A}{2}\left\{\sin ka\left[\cosh\varkappa l+\frac{\varkappa}{ik}\sinh\varkappa l\right]+\frac{k}{\varkappa}\cos ka\left[\sinh\varkappa l+\frac{\varkappa}{ik}\cosh\varkappa l\right]\right\}=1$$

$$\frac{A}{2}\left\{\sin ka\left[\cosh\varkappa l-\frac{\varkappa}{ik}\sinh\varkappa l\right]+\frac{k}{\varkappa}\cos ka\left[\sinh\varkappa l-\frac{\varkappa}{ik}\cosh\varkappa l\right]\right\}=C$$

Hence,

$$C=\frac{\sin ka\left[\cosh \varkappa l-\frac{\varkappa}{ik}\sinh \varkappa l\right]+\frac{k}{\varkappa}\cos ka\left[\sinh \varkappa l-\frac{\varkappa}{ik}\cosh \varkappa l\right]}{\sin ka\left[\cosh \varkappa l+\frac{\varkappa}{ik}\sinh \varkappa l\right]+\frac{k}{\varkappa}\cos ka\left[\sinh \varkappa l+\frac{\varkappa}{ik}\cosh \varkappa l\right]}$$

We note that the numerator is complex conjugate to the denominator. Hence, $|C|^2 = 1$. The incident (falling) wave is completely reflected at $x = 0$. We write the expression for A in the form

$$\frac{A}{4}e^{\varkappa l}\left\{\left(1+\frac{\varkappa}{ik}\right)\left(\sin ka+\frac{k}{\varkappa}\cos ka\right)\right.$$
$$\left.+\left(1-\frac{\varkappa}{ik}\right)\left(\sin ka-\frac{k}{\varkappa}\cos ka\right)e^{-2\varkappa l}\right\}=1$$

and we see that when $\varkappa$ is real ($E < V_0$), the second member is always much less than the first ($e^{-2\varkappa l} \ll 1$). So if we neglect it, we find that $|A|^2 \propto e^{-2\varkappa l}$, i.e. $|A|^2 \ll 1$ (the amplitude of the wave is much less in the inner region than in the outer). This holds for all values of energy except when

$$\sin k_0 a+\frac{k_0}{\varkappa_0}\cos k_0 a=0 \quad \text{or} \quad \tan k_0 a=-\frac{k_0}{\varkappa_0}$$

(If we compare this with Problem 45, we find that this condition gives the energy levels in a potential well of limited depth.) Then the member with $e^{-2\varkappa l}$ plays a decisive role and A grows substantially:

$$A=2i\frac{\sqrt{\varkappa_0^2+k_0^2}}{\varkappa_0-ik_0}e^{\varkappa_0 l} \quad \text{and} \quad C=-\frac{\varkappa_0+ik_0}{\varkappa_0-ik_0}$$

i.e.

$$|A|^2=4e^{2\varkappa_0 l} \quad \text{and} \quad |C|^2=1$$

Thus, near the energy eigenvalues for a particle in a potential well of finite depth the amplitude of the wave in the inner region changes abruptly from $e^{-2\varkappa_0 l}$ to $e^{+2\varkappa_0 l}$.

94. To characterize the forces that prevent the electrons from leaving the metal we place the origin of coordinates on the interface between the metal and vacuum and assume that the potential energy of the electrons in the metal is lower than their energy in the vacuum by V_0, i.e. we assume

$V = 0$ for $x < 0$ (in the metal) and $V = V_0$ for $x > 0$ (on the interface).

Let the x-axis be normal to the surface of the metal. If we apply the external electric field $\mathbf{E}$ in the positive direction of this axis, then for $x > 0$ the potential energy will be $V(x) = V_0 - e\,|\,\mathbf{E}\,|\,x$, and the probability of the electron passing through the barrier will be determined by the transmittance

$$D = D_0 \exp\left[-2\frac{\sqrt{2m}}{\hbar}\int_{x_1}^{x_2}\sqrt{V(x) - E_x}\,dx\right]$$

Now we need only compute the integral in the exponent.

The problem becomes one-dimensional. The only important factor is the movement along the x-axis, and $E_x = \frac{p_x^2}{2m}$ denotes the energy connected with this movement. Points x_1 and x_2 are determined from the condition $V(x_1) = V(x_2) = E_x$. For our problem $x_1 = 0$ and x_2 is determined by the equation

$$V_0 - e\,|\,\mathbf{E}\,|\,x_2 = E_x$$

Then

$$\int_0^{x_2}\sqrt{V_0 - E_x - e\,|\mathbf{E}|\,x}\,dx = -\frac{2}{3e\,|\mathbf{E}|}\left(V_0 - E_x - e\,|\mathbf{E}|\,x\right)^{3/2}\Big|_0^{x_2}$$

$$= \frac{2}{3e\,|\mathbf{E}|}\left(V_0 - E_x\right)^{3/2}$$

and

$$D = D_0 \exp\left[\frac{4\sqrt{2m}}{3\hbar e\,|\mathbf{E}|}\left(V_0 - E_x\right)^{3/2}\right]$$

i.e. grows as $|\,\mathbf{E}\,|$ and E_x grow.

If we denote dn as the number of electrons inside the metal (per unit volume) that possess momenta in the interval $(\mathbf{p}, \mathbf{p} + d\mathbf{p})$, the density of the electric current leaving the metal in the positive direction of the x-axis is

$$j = e\int v_x D\,dn, \quad \text{where} \quad v_x = \frac{p_x}{m}$$

The integration is done over all values of p_y and p_z and over $p_x > 0$. If we assume that the electron gas is extre-

mely degenerate (i.e. behaves as it would when $T = 0$ K), we get

$$dn = 2\,\frac{dp_x\,dp_y\,dp_z}{h^3}$$

(the mean occupation of a state is one) for $\frac{p^2}{2m} \leqslant \zeta$, where ζ is the maximum energy, i.e. the level of the chemical potential, and

$$dn = 0 \quad \text{for} \quad \frac{p^2}{2m} > \zeta$$

Hence, if we pass to cylindrical coordinates in momentum space and assume that $p_y = \rho\cos\varphi$ and $p_z = \rho\sin\varphi$, we can write

$$j = \frac{2e}{h^3}\int_0^{\sqrt{2m\zeta}} dp_x \int_0^{\sqrt{2m\zeta - p_x^2}} \rho\,d\rho \int_0^{2\pi} v_x D(v_x)\,d\varphi$$

Now we substitute $\eta = \zeta - E_x$, $d\eta = -v_x dp_x$, integrate over φ and ρ, and get

$$j = \frac{4\pi em}{h^3}\int_0^{\zeta} \eta\,D(\eta)\,d\eta$$

$$D(\eta) = D_0 \exp\left[-\frac{4}{3}\,\frac{\sqrt{2m}}{\hbar e\,|\mathbf{E}|}(V_0 - \zeta + \eta)^{3/2}\right]$$

Since $D(\eta)$ diminishes very quickly as η increases, the members with small η will play the dominant role. So if we expand the exponent in a power series in η and denote $\frac{2\sqrt{2m}}{\hbar e\,|\mathbf{E}|}(V_0 - \zeta)^{3/2} = q$ and $\frac{\eta}{V_0 - \zeta} = \xi$, we can extend the limits of integration over ξ to infinity and get

$$j = D_0\,\frac{4\pi em}{h^3}\,e^{-\frac{2}{3}q}(V_0 - \zeta)^2 \int_0^{\infty} e^{-q\xi}\xi\,d\xi = D\,\frac{4\pi em}{h^3}\,\frac{(V_0 - \zeta)^2}{q^2}\,e^{-\frac{2}{3}q}$$

$$= A\,|\mathbf{E}|^2 \exp\left[-\frac{4\sqrt{2m}}{3\,|\mathbf{E}|\,\hbar e}(V_0 - \zeta)^{3/2}\right].$$

95. The alpha-particle in the nucleus lies in a deep potential well. We may assume approximately that $V = -V_0$

for $r \leqslant r_0$; r_0 characterizes the range of nuclear forces. For $r \geqslant r_0$ (outside the nucleus), $V = \frac{2Ze^2}{r}$. The transmittance of the particle through a barrier limited by a straight line at $r = r_0$ and by a hyperbola $V = \frac{2Ze^2}{r}$ for $r > r_0$ is determined as

$$D = D_0 \exp\left(-\frac{2\sqrt{2m}}{\hbar}\int_{r_0}^{r_2}\sqrt{\frac{2Ze^2}{r} - E}\,dr\right) \tag{1}$$

where E is the energy of the particle falling on the barrier, and r_0 and r_2 are the points of retrogression, at which $V = E$, i.e. $r_2 = 2Ze^2/E$.

To calculate the integral in (1) we introduce $\cos^2 u = \frac{r}{r_2}$. Obviously, at $r = r_2$, we have $u = 0$. Now if we denote $\cos^2 u_0 = \frac{r_0}{r_2}$, we can write

$$I = \int_0^{u_0}\sqrt{\frac{1}{\cos^2 u} - 1} \times 2\sin u \cos u\, du \sqrt{E}\,\frac{2Ze^2}{E}$$

$$= \frac{2Ze^2}{\sqrt{E}}\left[u_0 - \frac{\sin 2u_0}{2}\right]$$

On the assumption that $\sqrt{\frac{r_0}{r_2}} \ll 1$ we expand u_0 in a series

$$u_0 = \arccos\sqrt{\frac{r_0}{r_2}} \approx \frac{\pi}{2} - \sqrt{\frac{r_0}{r_2}}$$

Then

$$\sin 2u_0 \approx 2\sqrt{\frac{r_0}{r_2}}$$

Hence

$$I \approx \frac{2Ze^2}{\sqrt{E}}\left[\frac{\pi}{2} - 2\sqrt{\frac{r_0}{r_2}}\right]$$

and

$$D \approx D_0 \exp\left[-\frac{4}{\hbar}\left(\frac{\pi Ze^2}{v_\infty} + \sqrt{2mZe^2 r_0}\right)\right]$$

where $v_\infty = \sqrt{2E/m}$ is the velocity of the emitted alpha particle measured far from the nucleus (where $V = 0$), and

$$\lambda = nD_0 \exp\left[-\frac{4}{\hbar}\left(\frac{\pi Z e^2}{v_\infty} + \sqrt{2mZe^2 r_0}\right)\right].$$

96. If we choose the z-axis in the direction of the vector of magnetic induction and write the components of the vector potential in the form $A_x = -By$, $A_y = A_z = 0$, we can reduce the equation

$$\frac{(\hat{\mathbf{p}} - e\mathbf{A})^2}{2m}\psi = E\psi$$

to the form

$$-\frac{\hbar^2}{2m}\Delta\psi - \frac{i\hbar}{m}eBy\frac{\partial\psi}{\partial x} + \frac{e^2}{2m}B^2y^2\psi = E\psi \qquad (1)$$

Since the coefficients do not depend on x or z, we can look for ψ in the form

$$\psi = e^{i\alpha x}e^{i\beta z}f(y)$$

After substituting ψ into (1), we get the equation for f:

$$-\frac{\hbar^2}{2m}\frac{d^2 f}{dy_1^2} + \frac{m\omega_0^2}{2}y_1^2 f = \varepsilon f$$

where we have denoted $\omega_0 = \frac{eB}{m}$, $\varepsilon = E_0 - \frac{\hbar^2\beta^2}{2m}$, $y_1 = y + \frac{\hbar\alpha}{eB}$.

We have come to the equation for the one-dimensional harmonic oscillator. If we introduce the dimensionless coordinate $\xi = y_1\sqrt{\frac{m\omega_0}{\hbar}}$, we can, in accordance with the solution of Problem 46, write

$$f_n(y_1) = Ce^{-\xi^2/2}H_n(\xi)$$

(H_n is the Hermite polynomial) and

$$\varepsilon_n = \left(n + \frac{1}{2}\right)\hbar\omega_0, \quad n = 0, 1, 2, \ldots$$

Thus, for an electron in a uniform magnetic field, the complete wave function and the energy spectrum are

$$\psi_{n\alpha\beta} = C_n e^{i(\alpha x + \beta z)}e^{-\xi^2/2}H_n(\xi)$$

$$E_{n\beta} = \frac{e\hbar}{2m}B(2n+1) + \frac{\hbar^2\beta^2}{2m}$$

The normalization condition yields

$$C_n = \frac{1}{\hbar} \sqrt[4]{\frac{m\omega_0}{\hbar\pi}} \frac{1}{\sqrt{2^n n!}}$$

The energy spectrum is continuous for motion along the z-axis (in the direction of the magnetic field) and discrete for motion in the plane of vector **B**.

97. If we compare the time-dependent equation for a particle in an electromagnetic field, which is characterized by a vector potential **A** and a scalar potential φ,

$$\frac{1}{2m}(-i\hbar\nabla - e\mathbf{A})^2\Psi = \left(i\hbar\frac{\partial}{\partial t} - e\varphi\right)\Psi$$

with the equation containing the changed potentials $\mathbf{A}' = \mathbf{A} + \operatorname{grad} f$ and $\varphi' = \varphi - \frac{\partial f}{\partial t}$ and, hence, the changed function Ψ',

$$\frac{1}{2m}(-i\hbar\nabla - e\mathbf{A} - e\operatorname{grad} f)^2\Psi' = \left(i\hbar\frac{\partial}{\partial t} - e\varphi + e\frac{\partial f}{\partial t}\right)\Psi'$$

we see that the operator ∇ on Ψ is equivalent to $\left(\nabla - \frac{ie}{\hbar}\operatorname{grad} f\right)$ on Ψ', and the operator $\frac{\partial}{\partial t}$ on Ψ is equivalent to $\left(\frac{\partial}{\partial t} - \frac{ie}{\hbar}\frac{\partial f}{\partial t}\right)$ on Ψ'. This shows that Ψ differs from Ψ' by a factor $e^{-ief/\hbar}$:

$$\Psi = \Psi' e^{-\frac{ief}{\hbar}}$$

Indeed, if we differentiate Ψ with respect to, say, t, we get

$$\frac{\partial\Psi}{\partial t} = \frac{\partial\Psi'}{\partial t} e^{-\frac{ief}{\hbar}} + \Psi'\frac{\partial e^{-\frac{ief}{\hbar}}}{\partial t} = e^{\frac{-ief}{\hbar}}\left(\frac{\partial}{\partial t} - \frac{ie}{\hbar}\frac{\partial f}{\partial t}\right)\Psi'$$

For this kind of transformation such expressions as $|\Psi|^2$ and $\langle\lambda\rangle = \int \Psi^*\hat{L}\Psi\, d\tau$ do not change.

98. We write the time-dependent equation and remove the parentheses $(\hat{\mathbf{p}} - e\mathbf{A})^2$, and we get

$$-\frac{\hbar^2}{2m}\Delta\Psi + \frac{i\hbar e}{m}(\mathbf{A}\cdot\operatorname{grad}\Psi) + \frac{i\hbar e}{2m}\Psi\operatorname{div}\mathbf{A} + \frac{e^2}{2m}\mathbf{A}^2\Psi + e\varphi\Psi = i\hbar\frac{\partial\Psi}{\partial t} \qquad (1)$$

The complex conjugate equation is

$$-\frac{\hbar^2}{2m}\Delta\Psi^* - \frac{i\hbar e}{m}(\mathbf{A}\cdot \text{ grad } \Psi^*) - \frac{i\hbar e}{2m}\Psi^* \operatorname{div}\mathbf{A} + \frac{e^2}{2m}\mathbf{A}^2\Psi^* + e\varphi\Psi^* = -i\hbar\frac{\partial\Psi^*}{\partial t} \tag{2}$$

From Eq. (1) multiplied by Ψ^* we subtract Eq. (2) multiplied by Ψ and get the expression for $\frac{\partial}{\partial t}|\Psi|^2$. Then we divide the difference by $i\hbar$ and get

$$\frac{\partial}{\partial t}|\Psi|^2 = \frac{i\hbar}{2m}(\Psi^*\Delta\Psi - \Psi\Delta\Psi^*) + \frac{e}{m}[\Psi^*(\mathbf{A}\cdot\text{grad }\Psi) + (\text{grad }\Psi^*\cdot\mathbf{A})\Psi + \Psi^*\Psi \operatorname{div}\mathbf{A}] \tag{3}$$

Since

$$\Psi^*\Delta\Psi - \Psi\Delta\Psi^* = \operatorname{div}(\Psi^* \text{ grad } \Psi - \Psi \text{ grad } \Psi^*)$$

and

$$\Psi^*(\mathbf{A}\cdot\text{grad }\Psi) + (\text{grad }\Psi^*\cdot\mathbf{A})\Psi + \Psi^*\Psi \operatorname{div}\mathbf{A} = \operatorname{div}(\mathbf{A}\Psi^*\Psi)$$

we can represent (3) as

$$\frac{\partial|\Psi|^2}{\partial t} + \operatorname{div}\mathbf{j} = 0$$

where

$$\mathbf{j} = \frac{i\hbar}{2m}(\Psi \text{ grad } \Psi^* - \Psi^* \text{ grad } \Psi) - \frac{e}{m}\mathbf{A}|\Psi|^2.$$

100. In the new representation we multiply the equation on the left by $\hat{S}^+$ and get

$$i\hbar\frac{\partial\psi}{\partial t} - \sum_k e_k\mathbf{r}_k\frac{\partial\mathbf{A}(t, 0)}{\partial t}\psi = \left[\sum_k\frac{\hat{S}^+\hat{\mathscr{P}}_k^2\hat{S}}{2m_k} + V\right]\psi \tag{1}$$

where

$$\hat{\mathscr{P}}_k = \hat{\mathbf{p}}_k - e_k\mathbf{A}(t, \mathbf{r}_k)$$

Obviously,

$$\hat{S}^+\hat{\mathscr{P}}_k^2\hat{S} = (\hat{S}^+\hat{\mathscr{P}}_k\hat{S})^2 \tag{2}$$

and

$$\hat{S}^+\hat{\mathscr{P}}_k\hat{S} = \hat{\mathbf{p}}_k - e_k[(\mathbf{r}_k\cdot\nabla_k)\mathbf{A}(t, \mathbf{r}_k)]_{\mathbf{r}_k=0} \tag{3}$$

Here we have carried out the expansion for $\mathbf{A}(t, \mathbf{r}_k)$ indicated in the problem. After substituting (3) into (2) and neglecting members of order r_k^2 and $(\mathbf{r}_k \cdot \mathbf{p}_k)$, we get in place of (1) the following equation for ψ:

$$i\hbar \frac{\partial \psi}{\partial t} = \sum_k \frac{\hat{\mathbf{p}}_k^2}{2m_k} \varphi + (V - \mathbf{Ed})\,\psi$$

where $\mathbf{d} = \sum\limits_k e_k \mathbf{r}_k$, $\mathbf{E} = -\frac{\partial \mathbf{A}(t, 0)}{\partial t}$ $(\operatorname{div} \mathbf{A} = 0)$.

101. It follows from $\hat{\sigma}_z \alpha = \alpha$ and $\hat{\sigma}_z \beta = -\beta$ that the operator $\hat{\sigma}_z$ has two eigenvalues equal to $+1$ and -1 and, hence, corresponds to a component of the spin vector (in units of $\hbar/2$). The eigenfunctions of $\hat{\sigma}_z$, i.e. α and β, will not, obviously, be the eigenfunctions of $\hat{\sigma}_x$ and $\hat{\sigma}_y$. But if we add and subtract the equations

$$\hat{\sigma}_x \alpha = \beta, \quad \hat{\sigma}_x \beta = \alpha, \tag{1}$$

$$\hat{\sigma}_y \alpha = i\beta, \quad \hat{\sigma}_y \beta = -i\alpha \tag{2}$$

we get

$$\hat{\sigma}_x (\alpha + \beta) = (\alpha + \beta), \qquad \hat{\sigma}_y (\alpha + i\beta) = \alpha + i\beta$$

$$\hat{\sigma}_x (\alpha - \beta) = -(\alpha - \beta), \quad \hat{\sigma}_y (\alpha - i\beta) = -(\alpha - i\beta)$$

These equations show that $\hat{\sigma}_x$ and $\hat{\sigma}_y$ have the same eigenvalues as $\hat{\sigma}_z$, equal to ± 1. If we form the expression

$$(\hat{\sigma}_x \hat{\sigma}_z - \hat{\sigma}_z \hat{\sigma}_x) \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \hat{\sigma}_x \begin{pmatrix} \alpha \\ -\beta \end{pmatrix} - \hat{\sigma}_z \begin{pmatrix} \beta \\ \alpha \end{pmatrix} = \begin{pmatrix} \beta \\ -\alpha \end{pmatrix} - \begin{pmatrix} -\beta \\ \alpha \end{pmatrix}$$

$$= \begin{pmatrix} 2\beta \\ -2\alpha \end{pmatrix} = -2i\hat{\sigma}_y \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

we see that $\hat{\sigma}_x \hat{\sigma}_z - \hat{\sigma}_z \hat{\sigma}_x = -2i\hat{\sigma}_y$. Aside from this, by applying operator $\hat{\sigma}_y$ to (1), we get

$$\hat{\sigma}_y \hat{\sigma}_x \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \hat{\sigma}_y \begin{pmatrix} \beta \\ \alpha \end{pmatrix} = \begin{pmatrix} -i\alpha \\ i\beta \end{pmatrix} = -i\sigma_z \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

It follows from this that

$$\hat{\sigma}_y \hat{\sigma}_x = -i\hat{\sigma}_z$$

The notation $\hat{\sigma}_x\hat{\sigma}_y \begin{pmatrix}\alpha\\ \beta\end{pmatrix} = i\sigma_z \begin{pmatrix}\alpha\\ \beta\end{pmatrix}$ means that

$$\hat{\sigma}_x\hat{\sigma}_y\alpha = i\hat{\sigma}_z\alpha \quad \text{and} \quad \hat{\sigma}_x\hat{\sigma}_y\beta = i\hat{\sigma}_z\beta$$

And since this holds for all the eigenfunctions (there are only two), it holds for any function. Hence, we can simply write the equation for the operators.

102. Since $\hat{\sigma}_x$ commutes with $\hat{\mathbf{p}}$ and $\hat{\mathbf{r}}$, if we compose $\frac{d\hat{\sigma}_x}{dt}$ following the general rule, we have

$$\frac{d\hat{\sigma}_x}{dt} = \frac{i}{\hbar}(\hat{H}\hat{\sigma}_x - \hat{\sigma}_x\hat{H}) = \frac{e}{m}(\hat{\sigma}_y B_z - \hat{\sigma}_z B_y).$$

103. We write the usual eigenvalue equation for the operator $\hat{\sigma}_x$: $\hat{\sigma}_x\chi = \lambda\chi$. Next we represent the sought function in the form of a matrix $\chi = \begin{pmatrix}a\\ b\end{pmatrix}$. If we act on this with the matrix $\hat{\sigma}_x = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$, we arrive at the equation

$$\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix} = \lambda\begin{pmatrix}a\\ b\end{pmatrix}$$

i.e.

$$\begin{pmatrix}b\\ a\end{pmatrix} = \lambda\begin{pmatrix}a\\ b\end{pmatrix} \quad \text{or} \quad b = \lambda a \quad \text{and} \quad a = \lambda b$$

and, hence, $\lambda^2 = 1$ or $\lambda = \pm 1$. Now we substitute the eigenvalue $\lambda_1 = 1$ into the equation $b = \lambda a$, and we get the eigenfunction $\chi_1 = a\begin{pmatrix}1\\ 1\end{pmatrix}$ that corresponds to this eigenvalue. If we do the same for $\lambda_2 = -1$, we get $\chi_2 = a\begin{pmatrix}1\\ -1\end{pmatrix}$. It follows from the normalization condition

$$\chi_1^+\chi_1 = |a|^2(1 \;\; 1)\begin{pmatrix}1\\ 1\end{pmatrix} = 2|a|^2 = 1$$

that $a = 2^{-1/2}$.

Similarly, for $\hat{\sigma}_y = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}$ from the equation

$$\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix} = \lambda\begin{pmatrix}a\\ b\end{pmatrix}$$

we get

$$-ib = \lambda a, \quad ia = \lambda b$$

Whence,

$$\lambda = \pm 1, \quad b = \pm ia; \quad \chi_{+1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}, \quad \chi_{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}.$$

104. We express the spin operator $\hat{\mathbf{S}}$ in terms of $\hat{\boldsymbol{\sigma}}$:

$$\hat{\mathbf{S}} = \frac{\hbar}{2}\hat{\boldsymbol{\sigma}}$$

where $\hat{\sigma}_x$, $\hat{\sigma}_y$, $\hat{\sigma}_z$ are the Pauli matrices, which satisfy the conditions $\hat{\sigma}_x^2 = \hat{\sigma}_y^2 = \hat{\sigma}_z^2 = 1$ and $\hat{\sigma}_x\hat{\sigma}_y = -\hat{\sigma}_y\hat{\sigma}_x$ (etc. cyclic). Now we project $\hat{\mathbf{S}}$ on a unit vector in the direction of $\mathbf{a}$, $\frac{(\hat{\mathbf{S}}\cdot\mathbf{a})}{a}$, and, squaring this expression and using the properties of $\hat{\sigma}_x$, $\hat{\sigma}_y$, $\hat{\sigma}_z$, we get

$$\frac{(\hat{\mathbf{S}}\cdot\mathbf{a})^2}{a^2} = \frac{\hbar^2}{4a^2}(\hat{\sigma}_x a_x + \hat{\sigma}_y a_y + \hat{\sigma}_z a_z)(\hat{\sigma}_x a_x + \hat{\sigma}_y a_y + \hat{\sigma}_z a_z)$$

$$= \frac{\hbar^2}{4a^2}[\hat{\sigma}_x^2 a_x^2 + \hat{\sigma}_y^2 a_y^2 + \hat{\sigma}_z^2 a_z^2 + (\hat{\sigma}_x\hat{\sigma}_y + \hat{\sigma}_y\hat{\sigma}_x)$$

$$\times a_x a_y + \ldots] = \frac{\hbar^2}{4}.$$

105. We remove the parentheses in the left-hand side of the relationship and then make use of the fact that $\hat{\sigma}_x^2 = \hat{\sigma}_y^2 = \hat{\sigma}_z^2 = 1$ and $\hat{\sigma}_x\hat{\sigma}_y = -\hat{\sigma}_y\hat{\sigma}_x = -i\hat{\sigma}_z$ (etc. cyclic). We find that $\hat{\sigma}_x\hat{A}_x\hat{\sigma}_x\hat{B}_x$ will reduce to $\hat{A}_x\hat{B}_x$ and

$$\hat{\sigma}_x\hat{A}_x\hat{\sigma}_y\hat{B}_y + \hat{\sigma}_y\hat{A}_y\hat{\sigma}_x\hat{B}_x = i\hat{\sigma}_z(\hat{A}_x\hat{B}_y - \hat{A}_y\hat{B}_x) = i\hat{\sigma}_z[\hat{\mathbf{A}}\times\hat{\mathbf{B}}]_z$$

Q.E.D.

106. If we use the commutation relations

$$\hat{\sigma}_x\hat{\sigma}_y - \hat{\sigma}_y\hat{\sigma}_x \equiv [\hat{\sigma}_x, \hat{\sigma}_y] = -[\hat{\sigma}_y, \hat{\sigma}_x] = 2i\hat{\sigma}_z$$

we can easily show that

$$\hat{\sigma}_+^2 = \hat{\sigma}_-^2 = 0, \quad [\hat{\sigma}_+, \hat{\sigma}_-] = \hat{\sigma}_z,$$

$$[\hat{\sigma}_\pm, \hat{\sigma}_x] = \pm\hat{\sigma}_z, \quad [\hat{\sigma}_\pm, \hat{\sigma}_y] = i\hat{\sigma}_z, \quad [\hat{\sigma}_\pm, \hat{\sigma}_z] = \mp\hat{\sigma}_\pm.$$

107. We prove (a), (b), and (c) by expanding $\sin(\hat{\sigma}_x\varphi)$ and $\cos(\hat{\sigma}_z\varphi)$ in series. The series contain for $\sin(\hat{\sigma}_x\varphi)$

none but odd powers and for $\cos(\hat{\sigma}_z\varphi)$ none but even powers. At the same time,

$$\hat{\sigma}_x^{2n+1}=\hat{\sigma}_x \quad \text{and} \quad \hat{\sigma}_z^{2n}=1$$

To prove (d) we take what we have proved in (c) and write

$$\hat{\sigma}_y e^{-i\hat{\sigma}_z\varphi}=\hat{\sigma}_y(\cos\varphi-i\hat{\sigma}_z\sin\varphi)=(\cos\varphi+i\hat{\sigma}_z\sin\varphi)\hat{\sigma}_y$$
$$=e^{i\hat{\sigma}_z\varphi}\hat{\sigma}_y.$$

108. We write $\hat{\sigma}_+\hat{\sigma}_-=\frac{1}{4}(\hat{\sigma}_x+i\hat{\sigma}_y)(\hat{\sigma}_x-i\hat{\sigma}_y)=\frac{1}{2}(1+\hat{\sigma}_z)$. If we square this expression, we find that

$$(\hat{\sigma}_+\hat{\sigma}_-)^2=\frac{1}{4}(1+\hat{\sigma}_z)^2=\frac{1}{2}(1+\hat{\sigma}_z)=\hat{\sigma}_+\hat{\sigma}_-$$

For an arbitrary n the relationship can be proved by mathematical induction.

109. By definition, $\hat{A}'=\hat{S}\hat{A}\hat{S}_+$, where $\hat{S}_+=e^{i\hat{\sigma}_z\varphi/2}$. Since $\hat{\sigma}_z$ anticommutes with $\hat{\sigma}_x$ and $\hat{\sigma}_y$, if we use the results of Problem 107 and the properties of the Pauli matrices ($\hat{\sigma}_x\hat{\sigma}_y=i\hat{\sigma}_z$, etc.), we can write

$$\hat{A}'=e^{-i\hat{\sigma}_z\varphi}\hat{A}=(\cos\varphi-i\hat{\sigma}_z\sin\varphi)(\hat{\sigma}_x\sin\varphi+\hat{\sigma}_y\cos\varphi)=\hat{\sigma}_y$$

In the same way

$$\hat{B}'=e^{-i\hat{\sigma}_z\varphi}\hat{B}=\hat{\sigma}_x.$$

110. Let α and β be functions that are acted upon by operators $\hat{\sigma}_x$, $\hat{\sigma}_y$, and $\hat{\sigma}_z$ in the way indicated in Problem 101. We denote function α for the neutron (n) by $\alpha_n\equiv\alpha(S_n)$, and for the proton ($p$) by $\alpha_p\equiv\alpha(S_p)$. Clearly, for a system of weakly interacting particles we must look for the function of the two-particle system in the form of a product of single-particle functions. Notably, we will look for the eigenfunction of the operator $\hat{S}_z=\hat{\sigma}_{nz}+\hat{\sigma}_{pz}$ in the form

$$\chi(S_n, S_p)=A\alpha_n\alpha_p+B\alpha_n\beta_p+C\beta_n\alpha_p+D\beta_n\beta_p$$

If we act on it with $\hat{S}_z$, we get

$$\hat{S}_z\chi=(\hat{\sigma}_{nz}+\hat{\sigma}_{pz})\chi=A\alpha_n\alpha_p+B\alpha_n\beta_p-C\beta_n\alpha_p-D\beta_n\beta_p$$
$$+A\alpha_n\alpha_p-B\alpha_n\beta_p+C\beta_n\alpha_p-D\beta_n\beta_p=2A\alpha_n\alpha_p-2D\beta_n\beta_p$$

Each of the four terms in χ therefore is an eigenfunction of $\hat{S}_z$, which correspond, respectively, to eigenvalues 2, 0, 0, —2. In units of $\hbar$ this corresponds to 1, 0, 0, —1. (The two functions $\alpha_n\beta_p$ and $\beta_n\alpha_p$ are degenerate.) Now we construct $\hat{\mathrm{S}}^2$ and determine its action on χ. Obviously,

$$\hat{\mathrm{S}}^2 = (\hat{\sigma}_{px} + \hat{\sigma}_{nx})^2 + \ldots = 6 + 2(\hat{\boldsymbol{\sigma}}_n \cdot \hat{\boldsymbol{\sigma}}_p), \text{ since } \hat{\sigma}_{px}^2 = \ldots = 1$$

and

$$\hat{\mathrm{S}}^2\chi = 6\chi + 2(\hat{\boldsymbol{\sigma}}_n \cdot \hat{\boldsymbol{\sigma}}_p)(A\alpha_n\alpha_p + B\alpha_n\beta_p + C\beta_n\alpha_p + D\beta_n\beta_p)$$

We find

$$A(\hat{\sigma}_{nx}\hat{\sigma}_{px} + \hat{\sigma}_{ny}\hat{\sigma}_{py} + \hat{\sigma}_{nz}\hat{\sigma}_{pz})\alpha_n\alpha_p$$
$$= A(\beta_n\beta_p - \beta_n\beta_p + \alpha_n\alpha_p) = A\alpha_n\alpha_p$$

In the same way,

$$B(\hat{\sigma}_{nx}\hat{\sigma}_{px} + \hat{\sigma}_{ny}\hat{\sigma}_{py} + \hat{\sigma}_{nz}\hat{\sigma}_{pz})\alpha_n\beta_p$$
$$= B[\beta_n\alpha_p + i\beta_n(-i\alpha_p) - \alpha_n\beta_p] = B(2\beta_n\alpha_p - \alpha_n\beta_p)$$

$$C(\hat{\sigma}_{nx}\hat{\sigma}_{px} + \hat{\sigma}_{ny}\hat{\sigma}_{py} + \hat{\sigma}_{nz}\hat{\sigma}_{pz})\beta_n\alpha_p$$
$$= C(\alpha_n\beta_p + \alpha_n\beta_p - \beta_n\alpha_p) = C(2\alpha_n\beta_p - \beta_n\alpha_p)$$

$$D(\hat{\sigma}_{nx}\hat{\sigma}_{px} + \hat{\sigma}_{ny}\hat{\sigma}_{py} + \hat{\sigma}_{nz}\hat{\sigma}_{pz})\beta_n\beta_p$$
$$= D(\alpha_n\alpha_p - \alpha_n\alpha_p + \beta_n\beta_p) = D\beta_n\beta_p$$

Consequently,

$$\hat{\mathrm{S}}^2\chi = 8A\alpha_n\alpha_p + 8D\beta_n\beta_p + 4(B + C)(\alpha_n\beta_p + \beta_n\alpha_p)$$

It follows from this that $\alpha_n\alpha_p$ and $\beta_n\beta_p$ are eigenfunctions of $\hat{\mathrm{S}}^2$ that correspond to the eigenvalue 8, i.e.

$$\left(\frac{\hbar}{2}\hat{\mathrm{S}}\right)^2 \alpha_n\alpha_p = 1(1+1)\hbar^2\alpha_n\alpha_p$$

and

$$\left(\frac{\hbar}{2}\hat{\mathrm{S}}\right)^2 \beta_n\beta_p = 1 \times 2 \times \hbar^2\beta_n\beta_p$$

Next we write the equation

$$\hat{\mathrm{S}}^2(B\alpha_n\beta_p + C\beta_n\alpha_p) = 4(B + C)(\alpha_n\beta_p + \beta_n\alpha_p)$$

where we must look for B and C such that $B\alpha_n\beta_p + C\beta_n\alpha_p$ be an eigenfunction of $\hat{\mathbf{S}}^2$. For this we write

$$4\,(B + C)\,(\alpha_n\beta_p + \beta_n\alpha_p) = 4\,\lambda(B\alpha_n\beta_p + C\beta_n\alpha_p)$$

Then

$$B + C = \lambda B, \qquad B + C = \lambda C$$

This system of two homogeneous linear equations has a non-trivial solution if the system determinant

$$\begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 1 = 0$$

Hence, there are two solutions:

(1) $\lambda = 2$, $B = C$. The eigenfunction $\alpha_n\beta_p + \beta_n\alpha_p$ corresponds to an eigenvalue of $\hat{\mathbf{S}}^2$ equal to 8 and an eigenvalue of $\hat{S}_z$ equal to 0;

(2) $\lambda = 0$, $B = -C$. The eigenfunction proves to be antisymmetric, i.e. equal to $\alpha_n\beta_p - \beta_n\alpha_p$. It corresponds to the zero eigenvalues of $\hat{\mathbf{S}}^2$ and $\hat{S}_z$.

Hence, the eigenvalue of $\left(\frac{\hbar}{2}\hat{\mathbf{S}}\right)^2$ equal to $2\hbar^2$ has corresponding to it three symmetric functions $\alpha_n\alpha_p$, $\beta_n\alpha_p + \alpha_n\beta_p$, $\beta_n\beta_p$, which describe states with eigenvalues of $\frac{\hbar}{2}\hat{S}_z$ equal to $\hbar$, 0, $-\hbar$. These states form a triplet. The eigenvalues of $\hat{\mathbf{S}}^2$ and $\hat{S}_z$ equal to zero have corresponding to them one antisymmetric function $\alpha_n\beta_p - \beta_n\alpha_p$, which is a singlet state.

111. Let $\hat{\mathbf{S}}_1 = \frac{\hbar}{2}\hat{\sigma}_1$ and $\hat{\mathbf{S}}_2 = \frac{\hbar}{2}\hat{\sigma}_2$ (where $\hat{\sigma}_i^2 = 3$, $\hat{\sigma}_{ix}^2 = \ldots = 1$). Let us consider the square of the sum of these operators:

$$\hat{\mathbf{S}}^2 = \hat{\mathbf{S}}_1^2 + \hat{\mathbf{S}}_2^2 + 2\,(\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{S}}_2)$$

In both the triplet and the singlet state (see the solution to Problem 110) $\hat{\mathbf{S}}^2$, $\hat{\mathbf{S}}_1^2$, and $\hat{\mathbf{S}}_2^2$ have definite eigenvalues: for $\hat{\mathbf{S}}^2$ this is $\hbar^2 s\,(s + 1)$ [$s = 1$ for the triplet state and $s = 0$ for the singlet state], and

$$\mathbf{S}_1^2 = \mathbf{S}_2^2 = \hbar^2 \frac{1}{2}\,\frac{3}{2} = \frac{3}{4}\,\hbar^2$$

Hence, we can write the values of the scalar product of the spin vector operators of two particles:

$$(\hat{S}_1 \cdot \hat{S}_2) \text{ whose eigenvalues are } \begin{cases} \dfrac{\hbar^2}{4} & \text{in the singlet state} \\ -\dfrac{3}{4}\hbar^2 & \text{in the triplet state} \end{cases}$$

In these states, respectively, $(\hat{\sigma}_1 \cdot \hat{\sigma}_2) = 1$ and $(\hat{\sigma}_1 \cdot \hat{\sigma}_2) = -3$.

112. In accordance with the solution of Problem 111 we denote the eigenfunctions of the triplet and singlet states as χ_t and χ_s and write

$$(\hat{\sigma}_1 \cdot \hat{\sigma}_2)\,\chi_t = 1\chi_t; \quad (\hat{\sigma}_1 \cdot \hat{\sigma}_2)\,\chi_s = -3\chi_s$$

Since the three triplet and one singlet spin functions form a complete set for the system of two particles, we must first investigate the action of $(\hat{\sigma}_1 \cdot \hat{\sigma}_2)^k$ on χ_t and χ_s. Obviously,

$$(\hat{\sigma}_1 \cdot \hat{\sigma}_2)^2\,\chi_t = (\hat{\sigma}_1 \cdot \hat{\sigma}_2)\,(\hat{\sigma}_1 \cdot \hat{\sigma}_2)\,\chi_t = 1^2\chi_t$$

therefore,

$$(\hat{\sigma}_1 \cdot \hat{\sigma}_2)^k\,\chi_t = 1^k\chi_t = \chi_t$$

In the same way,

$$(\hat{\sigma}_1 \cdot \hat{\sigma}_2)^2\,\chi_s = (\hat{\sigma}_1 \cdot \hat{\sigma}_1)\,(-3\chi_s) = (-3)^2\,\chi_s$$

and

$$(\hat{\sigma}_1 \cdot \hat{\sigma}_2)^k\,\chi_s = (-3)^k\,\chi_s$$

If we assume that

$$(\hat{\sigma}_1 \cdot \hat{\sigma}_2)^k = A + B\,(\hat{\sigma}_1 \cdot \hat{\sigma}_2)$$

we can choose A and B in such a way that the relation holds when acting on χ_t and χ_s and, hence, on any spin function of the two particles. Since

$$(\hat{\sigma}_1 \cdot \hat{\sigma}_2)^k\,\chi_t = (A + B)\,\chi_t \quad \text{and} \quad (\hat{\sigma}_1 \cdot \hat{\sigma}_2)^k\,\chi_t = 1\chi_t$$

we have

$$A + B = 1$$

On the other hand,

$$(\hat{\sigma}_1 \cdot \hat{\sigma}_2)^k\,\chi_s = [A + B\,(\hat{\sigma}_1 \cdot \hat{\sigma}_2)]\,\chi_s = (A - 3B)\,\chi_s$$

and

$$(\hat{\sigma}_1 \cdot \hat{\sigma}_2)^k \chi_s = (-3)^k \chi_s$$

i.e.

$$A - 3B = (-3)^k$$

It follows from this that

$$A = \frac{3+(-3)^k}{4}, \quad B = \frac{1-(-3)^k}{4}$$

Hence,

$$(\hat{\sigma}_1 \cdot \hat{\sigma}_2)^k = \frac{3+(-3)^k}{4} + \frac{1-(-3)^k}{4} (\hat{\sigma}_1 \cdot \hat{\sigma}_2)$$

For example, for $k=2$

$$(\hat{\sigma}_1 \cdot \hat{\sigma}_2)^2 = 3 - 2(\hat{\sigma}_1 \cdot \hat{\sigma}_2).$$

113. We use the matrix representation to solve this problem. Let us expand the eigenfunction in a complete set of states with $S = 1$:

$$\psi = \sum_{m=-1}^{+1} a_m \psi_{1m} \tag{1}$$

The cefficients a_m satisfy the equations

$$\sum_{m'=-1}^{+1} H_{1m,\, 1m'} a_{m'} = \varepsilon a_m \tag{2}$$

We reorganize the Hamiltonian to produce

$$\hat{H} = \frac{1}{4}(A-B)(\hat{S}_+^2 + \hat{S}_-^2) + \left(C - \frac{A+B}{2}\right)\hat{S}_z^2 + A + B$$

where $\hat{S}_{\pm} = \hat{S}_x \pm i\hat{S}_y$, and the eigenvalue of $\hat{\mathbf{S}}^2$ is assumed to be 2.

To calculate the matrix elements of $\hat{S}_+^2$ and $\hat{S}_-^2$ we consider the system with spin 1 as a pair of particles whose spins are equal to 1/2. Then, as in the solution to Problem 110, we denote

$$\psi_{11} = \alpha_1\alpha_2, \quad \psi_{10} = \frac{1}{\sqrt{2}}(\alpha_1\beta_2 + \alpha_2\beta_1), \quad \psi_{1-1} = \beta_1\beta_2$$

(We use 1 and 2 instead of the symbols n and p.)

Clearly, the spin vector operator of the pair of particles in units of $\hbar$ will be

$$\hat{\mathbf{S}}_x = \frac{1}{2}(\hat{\sigma}_{x1} + \hat{\sigma}_{x2}) \quad (x = x,\, y,\, z)$$

Using the rules indicated in Problem 101, we see that

$$\hat{S}_x\psi_{11} = \frac{1}{\sqrt{2}}\psi_{10}, \qquad \hat{S}_y\psi_{11} = \frac{i\sqrt{2}}{2}\psi_{10}$$

$$\hat{S}_x\psi_{10} = \frac{\sqrt{2}}{2}(\psi_{1\,-1} + \psi_{11}), \quad \hat{S}_y\psi_{10} = \frac{i\sqrt{2}}{2}(\psi_{1\,-1} - \psi_{11})$$

$$\hat{S}_x\psi_{1\,-1} = \frac{1}{\sqrt{2}}\psi_{10}, \qquad \hat{S}_y\psi_{1\,-1} = \frac{-i}{\sqrt{2}}\psi_{10}$$

Now it is easy to find the result of $\hat{S}_\pm^2$ acting on all three functions. We see that

$$\hat{S}_+\psi_{1\,-1} = \sqrt{2}\,\psi_{10} \quad \text{and} \quad \hat{S}_+^2\psi_{1\,-1} = 2\psi_{11}$$

$$\hat{S}_-\psi_{11} = \sqrt{2}\,\psi_{10} \quad \text{and} \quad \hat{S}_-^2\psi_{11} = 2\psi_{1\,-1}$$

The action of $\hat{S}_\pm^2$ on the other functions yields zero.

If we substitute the values $(\hat{S}_\pm^2)_{mm'}$ into Eqs. (2) and recall that $\hat{S}_z^2$ has none but diagonal elements, we arrive at the following equations:

$$\left.\begin{aligned} \left(C + \frac{A+B}{2} - \varepsilon\right)a_1 + \frac{A-B}{2}a_{-1} = 0 \\ (A+B-\varepsilon)\,a_0 = 0 \\ \frac{A-B}{2}a_1 + \left(C + \frac{A+B}{2} - \varepsilon\right)a_{-1} = 0 \end{aligned}\right\} \tag{3}$$

We equate the system determinant with zero and get the energy levels and [using Eqs. (2) and (3)] the corresponding spin functions:

$$\varepsilon_1 = C + B, \quad \psi_1 = \frac{1}{\sqrt{2}}(\psi_{1\,-1} - \psi_{11})$$

$$\varepsilon_2 = A + B, \quad \psi_2 = \psi_{10}$$

$$\varepsilon_3 = A + C, \quad \psi_3 = \frac{1}{\sqrt{2}}(\psi_{11} + \psi_{1\,-1}).$$

114. Before the electron enters the auxiliary field $\mathbf{B}'$, it is described by the wave function $\Psi = e^{i(ky-\omega t)}\,\overline{\alpha}(\sigma)$ that satisfies the equation

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\Psi + \mu_B B\hat{\sigma}_z\Psi \quad \left(\mu_B = -\frac{e\hbar}{2mc}\right)$$

where $\hbar\omega = \dfrac{\hbar^2k^2}{2m} + \mu_B B$.

If now, at $t = 0$, the field $\mathbf{B}'$ is switched on, the equation takes the form

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\Psi + \mu_B(B\hat{\sigma}_z + B'\hat{\sigma}_x)\Psi$$

Now we substitute

$$\Psi = e^{i(ky-\omega_0 t)}[a(t)\alpha(\sigma) + b(t)\beta(\sigma)]$$

where $\hbar\omega_0 = \dfrac{\hbar^2k^2}{2m}$, and find the equations for $a(t)$ and $b(t)$:

$$i\hbar\frac{da}{dt} = \mu_B Ba + \mu_B B'b$$

$$i\hbar\frac{db}{dt} = -\mu_B Bb + \mu_B B'a$$

For their solution we assume that

$$a(t) = Ae^{-i\varepsilon t/\hbar}, \quad b(t) = Ce^{-i\varepsilon t/\hbar}$$

and for A and C we get a system of two algebraic equations:

$$(\mu_B B - \varepsilon)A + \mu_B B'C = 0$$
$$\mu_B B'A - (\mu_B B + \varepsilon)C = 0$$

which has a nontrivial solution if the system determinant is zero. The determinant vanishes for two different values of ω,

$$\varepsilon_1 = \mu_B\sqrt{B^2+B'^2} \quad \text{and} \quad \varepsilon_2 = -\mu_B\sqrt{B^2+B'^2}$$

Also

$$C_1 = A_1\frac{-B+\sqrt{B^2+B'^2}}{B'} \quad \text{and} \quad C_2 = A_2\frac{-B-\sqrt{B^2+B'^2}}{B'}$$

The general solution is

$$\Psi = e^{i(ky-\omega_0 t)}\left\{A_1e^{-i\varepsilon_1 t/\hbar}\left[\alpha(\sigma) + \frac{-B+\sqrt{B^2+B'^2}}{B'}\beta(\sigma)\right]\right.$$
$$\left. + A_2e^{-i\varepsilon_2 t/\hbar}\left[\alpha(\sigma) - \frac{B+\sqrt{B^2+B'^2}}{B'}\beta(\sigma)\right]\right\} \quad (1)$$

There still remain two integration constants, A_1 and A_2, to be determined from the initial conditions $a(0) = 1$ and $b(0) = 0$. This yields $A_1 + A_2 = 1$ and $C_1 + C_2 = 0$. We substitute the expressions for A_1 and A_2 into (1), denote $\tau = \frac{\mu_B}{\hbar}\sqrt{B^2 + B'^2}$, and get the following equation

$$\psi = e^{i(ky-\omega_0 t)}\left\{\left[\cos\tau - i\frac{B}{\sqrt{B^2+B'^2}}\sin\tau\right]\alpha(\sigma) + i\frac{B'}{\sqrt{B^2+B'^2}}\sin\tau\,\beta(\sigma)\right\}$$

Clearly, the probability of spin flip, i.e. of finding the particle with spin downward at the time $t_0 = \frac{l}{v}$, is determined by the square of the modulus of the coefficient of $\beta(\sigma)$ and is equal to

$$\frac{B'^2}{B^2+B'^2}\sin^2\tau$$

with a maximum at $\tau = \frac{\pi}{2} = \frac{\mu_B}{\hbar}\sqrt{B^2+B'^2}\frac{l}{v}$.

115. Since the solution of the unperturbed problem for $\varepsilon_1 = \varepsilon_2 = 0$ (the simple harmonic oscillator; see Problem 46) gives us the nondegenerate eigenvalues of $\hat{H}_0$,

$$E_n^0 = \left(n + \frac{1}{2}\right)\hbar\omega$$

to find the eigenvalues and eigenfunctions of operator $\hat{H} = \hat{H}_0 + \hat{W}$ we must use the formulas

$$E_n = E_n^0 + \langle n | W | k\rangle + \sum_{k\neq n}\frac{|\langle n | W | k\rangle|^2}{E_n^0 - E_k^0} \tag{1}$$

and

$$\psi_n = \psi_n^0 + \sum_{k\neq n}\frac{\langle k | W | n\rangle}{E_n^0 - E_k^0}\psi_k^0 \tag{2}$$

where $\psi_n^0 = C_n e^{-\xi^2/2}H_n(\xi)$, H_n is the Hermite polynomial, $n = 0, 1, 2, \ldots$, and $\xi = x(m\omega/\hbar)^{1/2}$.

The problem reduces to calculating the matrix elements

$$\langle k | W | n\rangle = \langle k | W_1 | n\rangle + \langle k | W_2 | n\rangle$$

where $W_1 = \varepsilon_1 x^3$ and $W_2 = \varepsilon_2 x^4$. If we use the matrix elements of the coordinate calculated in Problem 74

$$\langle n|x|n+1\rangle = \sqrt{\frac{\hbar(n+1)}{2m\omega}} \quad \text{and} \quad \langle n|x|n-1\rangle = \sqrt{\frac{\hbar n}{2m\omega}}$$

(the other $\langle n \mid x \mid k\rangle$ are zero) and write these results in a compact form

$$\langle n|x|k\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{n}\,\delta_{n,\,k+1} + \sqrt{n+1}\,\delta_{k,\,n+1}\right)$$

we can evaluate $\langle n|W_i|k\rangle$ by using the law of matrix multiplication:

$$\langle n|W_1|k\rangle = \varepsilon_1\langle n|x^3|k\rangle = \varepsilon_1\sum_l \langle n|x^2|l\rangle\langle l|x|k\rangle$$

In turn,

$$\langle n|x^2|l\rangle = \sum_p \langle n|x|p\rangle\langle p|x|l\rangle =$$

$$= \frac{\hbar}{2m\omega}\sum_p\left(\sqrt{n}\,\delta_{n,\,p+1} + \sqrt{n+1}\,\delta_{n+1,\,p}\right)$$

$$\times\left(\sqrt{p}\,\delta_{p,\,l+1} + \sqrt{p+1}\,\delta_{l,\,p+1}\right)$$

Obviously, in products of type $\sqrt{p}\,\delta_{p,\,l+1}$ we can replace $\sqrt{p}$ by $\sqrt{l+1}$ and $\delta_{n,\,p+1}$ by $\delta_{p,\,n-1}$. Resultantly,

$$\langle n|x^2|l\rangle = \frac{\hbar}{2m\omega}\Big[\sqrt{n(l+1)}\sum_p \delta_{n-1,\,p}\delta_{p,\,l+1}$$

$$+\sqrt{(n+1)(l+1)}\sum_p \delta_{n+1,\,p}\delta_{p,\,l+1}$$

$$+\sqrt{nl}\sum_p \delta_{n-1,\,p}\delta_{p,\,l-1} + \sqrt{(n+1)\,l}\sum_p \delta_{n+1,\,p}\delta_{p,\,l-1}\Big]$$

and, for example, the first term will be

$$\sqrt{n(l+1)}\sum_p \delta_{n-1,\,p}\delta_{p,\,l+1} = \sqrt{n(l+1)}\,(\delta\cdot\delta)_{n-1,\,l+1}$$

$$= \delta_{n,\,l+2}\sqrt{n(l+1)} = \delta_{n,\,l+2}\sqrt{n(n-1)}$$

Since δ_{ik} is a unit matrix, $\delta^2 = \delta$.

When we have performed this operation with all terms, we get

$$\langle n | x^2 | l\rangle = \left(\frac{\hbar}{2m\omega}\right) [\sqrt{n(n-1)}\,\delta_{n,\,l+2} + (2n+1)\,\delta_{nl} + \sqrt{(n+1)(n+2)}\,\delta_{n,\,l-2}]$$

If we substitute this expression into $\langle n \mid x^3 \mid k\rangle$ and make similar calculations, we can write

$$\langle n | x^3 | k\rangle = \left(\frac{\hbar}{2m\omega}\right)^{3/2} \sum_l [\sqrt{n(n-1)}\,\delta_{n,\,l+2} + (2n+1)\,\delta_{nl} + \sqrt{(n+1)(n+2)}\,\delta_{n,\,l-2}]\,[\sqrt{l}\,\delta_{l,\,k+1} + \sqrt{k}\,\delta_{l,\,k-1}]$$
$$= \left(\frac{\hbar}{2m\omega}\right)^{3/2} [\sqrt{n(n-1)(n-2)}\,\delta_{n,\,k+3} + 3n^{3/2}\delta_{n,\,k+1} + 3(n+1)^{3/2}\,\delta_{n,\,k-1} + \sqrt{(n+1)(n+2)(n+3)}\,\delta_{n,\,k-3}]$$

Hence, $\langle n \mid W_1 \mid n\rangle = 0$, and the matrix element $\langle n \mid W_1 \mid k\rangle$ for a given n is nonzero in only four cases: at $k = n \pm 1$ and at $k = n \pm 3$. Consequently, this term gives a correction to (1) in none but the second approximation, and to (2) in the first, and we can limit ourselves to these. If we bear in mind that the denominators in formulas (1) and (2) for $k = n \pm 1$ turn into $E_n^0 - E_{n\pm 1}^0 = \mp\hbar\omega$, and for $k = n \pm 3$ into $E_n^0 - E_{n\pm 3}^0 = \mp 3\hbar\omega$, we can write

$$\psi_n = \psi_n^0 + \left(\frac{\hbar}{2m\omega}\right)^{3/2} \left[\frac{\sqrt{n(n-1)(n-2)}}{3\hbar\omega}\,\psi_{n-3}^0 + \frac{\sqrt{(n+1)(n+2)(n+3)}}{-3\hbar\omega}\,\psi_{n+3}^0 + \frac{3(n+1)^{3/2}}{-\hbar\omega}\,\psi_{n+1}^0 + \frac{3n^{3/2}}{\hbar\omega}\,\psi_{n-1}^0\right]$$

When we calculate the energy, we must consider not only the correction of the second order in relation to $\hat{W}_1$ but also the correction of the first order in relation to $\hat{W}_2$, i.e. we must calculate $\langle n \mid W_2 \mid n\rangle = \varepsilon_2 \langle n \mid x^4 \mid n\rangle$. In a similar

way we get

$$\langle n|x^4|n\rangle = \sum_k \langle n|x^3|k\rangle\langle k|x|n\rangle$$

$$= \left(\frac{\hbar}{2m\omega}\right)^2 \sum_k [\sqrt{n(n-1)(n-2)}\,\delta_{n,k+3} + 3n^{3/2}\delta_{n,k+1}$$

$$+ 3(n+1)^{3/2}\delta_{n,k-1} + \sqrt{(n+1)(n+2)(n+3)}\,\delta_{n,k-3}]$$

$$\times [\sqrt{n}\,\delta_{n,k+1} + \sqrt{n+1}\,\delta_{n,k-1}]$$

$$= \left(\frac{\hbar}{2m\omega}\right)^2 [3n^2 + 3(n+1)^2]$$

(We get results that differ from zero only when we multiply $\delta_{n,k+1}$ by $\delta_{n,k+1}$ and $\delta_{n,k-1}$ by $\delta_{n,k-1}$. The other products give zero; for example $\sum_k \delta_{n,k+3}\times\delta_{n,k+1} = \delta_{n-3,n-1} = 0$.)

Finally,

$$E_n = \left(n+\frac{1}{2}\right)\hbar\omega + \varepsilon_2\langle n|x^4|n\rangle + \varepsilon_1^2\left[\frac{(\langle n|x^3|n-3\rangle)^2}{3\hbar\omega} + \frac{(\langle n|x^3|n+3\rangle)^2}{-3\hbar\omega} + \frac{(\langle n|x^3|n-1\rangle)^2}{\hbar\omega} + \frac{(\langle n|x^3|n+1\rangle)^2}{-\hbar\omega}\right]$$

and after we substitute the evaluated matrix elements, we find that

$$E_n = \left(n+\frac{1}{2}\right)\hbar\omega + 3\varepsilon_2\left(\frac{\hbar}{2m\omega}\right)^2(2n^2+2n+1) - \frac{\varepsilon_1^2}{\hbar\omega}\left(\frac{\hbar}{2m\omega}\right)^3(30n^2+30n+11).$$

116. In the absence of a magnetic field the unperturbed equation

$$\left[-\frac{\hbar^2}{2\mu}\Delta + V(r)\right]\psi = E_{nl}\psi$$

has the solution

$$\psi = \psi_{nlm}(r,\theta,\varphi) = R_{nl}(r)\,P_{lm}(\cos\theta)\,e^{im\varphi}$$

In the presence of a magnetic field the Hamiltonian is $\hat{H} = \hat{H}_0 + \frac{i\hbar e}{\mu}(\mathbf{A}\cdot\nabla)$ [if we neglect $\mathbf{A}^2$]; here μ is the mass of the particle. Considering the second term to be

the energy of perturbation, we can find the correction to E_{nl} in the first approximation:

$$E'_{nl} = E - E_{nl} = \langle n \mid W \mid n \rangle = \int \psi^*_{nlm} \frac{i\hbar e}{\mu} (\mathbf{A} \cdot \nabla) \psi_{nlm} \, d\tau$$

If the z-axis is directed along the magnetic induction vector $\mathbf{B}$, we can choose the vector potential as

$$A_x = -\frac{1}{2} By, \quad A_y = \frac{1}{2} Bx, \quad A_z = 0$$

Then

$$i\hbar (\mathbf{A} \cdot \nabla) = \frac{i\hbar B}{2} \left(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right) = \frac{i\hbar B}{2} \frac{\partial}{\partial \varphi}$$

and since $\frac{\partial \psi_{nlm}}{\partial \varphi} = im\psi_{nlm}$ $\left(\psi_{nlm} \right.$ is an eigenfunction of the operator $\left. \hat{L}_z = i\hbar \frac{\partial}{\partial \varphi} \right)$, the correction to the energy is

$$E'_{nl} = -\frac{eB}{2\mu} \hbar m \int | \psi_{nlm} |^2 \, d\tau = -\frac{eB}{2\mu} \hbar m$$

To determine the eigenfunction of the perturbed problem and the corrections to the energy in the second approximation we must calculate the nondiagonal matrix elements

$$\langle nlm \mid W \mid nlm' \rangle = \frac{i\hbar Be}{2\mu} \int \psi^*_{nlm} \frac{\partial \psi_{nlm'}}{\partial \varphi} \, d\tau$$
$$= -\frac{\hbar Be}{2\mu} m' \int \psi^*_{nlm} \psi_{nlm'} \, d\tau$$

which are equal to zero if $m \neq m'$.

Thus, a magnetic field does not change the eigenfunctions of the unperturbed problem ($\psi'_{nlm} = \psi^0_{nlm}$), and the energy correction in the second approximation is zero. The energy levels split depending on m:

$$E_{nlm} = E_{nl} - \frac{eB}{2\mu} \hbar m.$$

117. Once we have calculated the matrix elements of the perturbation $\hat{W} = e \mid \mathbf{E} \mid a \cos \varphi$ using the functions of the unperturbed problem, $U^0_m = \frac{1}{\sqrt{2\pi}} e^{im\varphi}$, we find that only

$$\langle m \mid W \mid m-1 \rangle = \langle m \mid W \mid m+1 \rangle = \frac{ea \mid \mathbf{E} \mid}{2}$$

are nonzero and, hence,

$$U_m = \frac{1}{\sqrt{2\pi}} e^{im\varphi} - \frac{\mu a^2 e\,|\mathbf{E}|\,a}{\hbar^2 \sqrt{2\pi}} \left[\frac{e^{i(m-1)\varphi}}{2m-1} - \frac{e^{i(m+1)\varphi}}{2m+1} \right]$$

$$E_m = \frac{\hbar^2 m^2}{2\mu a^2} + \frac{e\,|\mathbf{E}|^2 a^4}{\hbar^2} \mu \frac{1}{4m^2-1}.$$

118. The solution of the unperturbed problem is $\psi_k^0 = \frac{1}{\sqrt{L^3}} e^{i\mathbf{kr}}$, and $E_k^0 = \frac{\hbar^2 k^2}{2m}$. To make the spectrum discrete we impose the requirement that

$$\psi_k^0 (x+L) = \psi_k^0 (x)$$

To calculate the matrix elements of the energy of perturbation we expand it in the Fourier series:

$$V(\mathbf{r}) = \sum_{\mathbf{g}} V_{\mathbf{g}} e^{2\pi i \mathbf{gr}}$$

Then

$$V_{\mathbf{k}'\mathbf{k}} = \sum_{\mathbf{g}} V_{\mathbf{g}} \frac{1}{L^3} \int_{L^3} e^{i(\mathbf{k}-\mathbf{k}'+2\pi\mathbf{g})\mathbf{r}} d\tau = V_{\mathbf{g}_0}$$

where $2\pi\mathbf{g}_0 = \mathbf{k}' - \mathbf{k}$, and we have the following solution of the perturbed problem

$$\psi_{\mathbf{k}} = \frac{1}{\sqrt{L^3}} e^{i\mathbf{kr}} + \frac{2m}{\hbar^2 \sqrt{L^3}} \sum_{\mathbf{g} \neq 0} \frac{V_{\mathbf{g}} e^{i(\mathbf{k}+2\pi\mathbf{g})\mathbf{r}}}{\mathbf{k}^2 - (\mathbf{k}+2\pi\mathbf{g})^2}$$

and

$$E_k = \frac{\hbar^2 k^2}{2m} + V_0 + \frac{2m}{\hbar^2} \sum_{\mathbf{g} \neq 0} \frac{|V_{\mathbf{g}}|^2}{\mathbf{k}^2 - (\mathbf{k}+2\pi\mathbf{g})^2}$$

If $E_k^0 = E_{\mathbf{k}+2\pi\mathbf{g}}$, we must seek the solution of the perturbed problem in the form

$$\psi = C_1 \psi_{\mathbf{k}}^0 + C_2 \psi_{\mathbf{k}+2\pi\mathbf{g}}^0$$

From the secular equation we get

$$\psi^{1,2} = \frac{1}{\sqrt{2L^3}} [e^{i\mathbf{kr}} \pm e^{i(\mathbf{k}+2\pi\mathbf{g})\mathbf{r}}] \qquad E_k^{1,2} = \frac{\hbar^2 k^2}{2m} + V_0 \pm |V_{\mathbf{g}}|$$

The energy spectrum of the electron in the periodic field has forbidden bands near $(\mathbf{k}\cdot\mathbf{g}) = -g^2$, the width of which is $2\,|V_{\mathbf{g}}|$.

119. In the absence of the field the unperturbed problem [the hydrogen problem; see (III-24) and (III-24a)] has n^2-fold degenerate eigenvalues. Hence, the value $n = 2$ has corresponding to it four eigenfunctions determined by the equation

$$\hat{H}_0 \psi_{2lm} = E_2^0 \psi_{2lm}$$

We write them out in more detail:

$$\psi_1^0 = \psi_{200} = R_{20}(r), \qquad \psi_2^0 = \psi_{210} = R_{21}(r) P_{10}(\cos\theta) = R_{21}\cos\theta$$
$$\psi_3^0 = \psi_{211} = R_{21}(r) P_{11}(\cos\theta) e^{i\varphi} = R_{21}\sin\theta e^{i\varphi}$$
$$\psi_4^0 = \psi_{21-1} = R_{21}\sin\theta e^{-i\varphi} \qquad (1)$$

Directing the z-axis along the external electric field $\mathbf{E}$, in which the atom is placed, we can write the Hamiltonian of the perturbed problem in the form

$$\hat{H} = \hat{H}^0 - e|\mathbf{E}|z, \quad \text{obviously,} \quad \hat{W} = -e|\mathbf{E}|z$$

The eigenfunctions of this operator that correspond to E_2^0 must be sought in the form of a linear combination of functions (1): $\psi = \sum_{i=1}^{4} c_i \psi_i^0$. The energy levels are found from the condition that the system determinant be zero:

$$|(E_2^0 - E)\delta_{ik} + \langle i | W | k \rangle| = 0 \quad (i, k = 1, 2, 3, 4) \quad (2)$$

Clearly, $\langle i | W | k \rangle$ can be represented as a product of integrals over r, θ, and φ:

$$\langle i | W | k \rangle = -e|\mathbf{E}| \int \psi_i^{0*} z \psi_k^0 \, d\tau = -e|\mathbf{E}| I_r I_\theta I_\varphi$$

The integral over φ can be evaluated thus:

$$I_\varphi = \int_0^{2\pi} e^{-im\varphi} e^{im'\varphi} \, d\varphi = 0 \quad \text{for} \quad m \neq m'$$

Hence,

$$\langle 1 | W | 3 \rangle = \langle 1 | W | 4 \rangle = \langle 2 | W | 3 \rangle = \langle 2 | W | 4 \rangle = \langle 3 | W | 4 \rangle = 0$$

In addition, if $m = m'$ and $l = l'$, we get

$$I_\theta = \int_0^\pi P_{lm}(\cos\theta)\cos\theta P_{lm}(\cos\theta)\sin\theta\, d\theta$$
$$= \int_{-1}^{1} |P_{lm}(x)|^2 x\, dx = 0$$

because $|P_{lm}(x)|^2$ is an even function of x and, hence, the integrand is odd. For the same reason all the $\langle k \mid W \mid k\rangle$ are zeros. Only $\langle 1 \mid W \mid 2\rangle$ and $\langle 2 \mid W \mid 1\rangle$ are nonzero:

$$\langle 1 | W | 2\rangle = -e|\mathbf{E}| \int_0^\infty R_{20}R_{21}r^3\, dr \int_0^\pi \cos^2\theta \sin\theta\, d\theta \int_0^{2\pi} d\varphi$$
$$= -\frac{4\pi}{3} e|\mathbf{E}| I_r$$

To calculate $I_r = \int_0^\infty R_{20}R_{21}r^3\, dr$ we must write the normalized radial functions for hydrogen explicitly. According to (III-24),

$$R_{nl} = e^{-\frac{r}{na}} \sum_{k=l}^{n-1} a_k \left(\frac{r}{a}\right)^k$$

where $a = \frac{\hbar^2}{\mu e^2}$ is the Bohr radius, and different a_k are linked by the recurrence relation

$$a_k = \frac{2\left(\frac{k}{n} - 1\right) a_{k-1}}{k(k+1) - l(l+1)}$$

Consequently, $R_{21} = a_1 e^{-r/2a} \frac{r}{a}$ $\left(\text{in this case } l = n - 1 = 1 \text{ and the entire sum reduces to one term } a_1 \frac{r}{a}\right)$ and $R_{20} = a_0' e^{-r/2a} \times \left(1 - \frac{r}{2a}\right)$. $\left(\text{Here } n - 1 = 1,\ l = 0, \text{ and } a_1 = \frac{2\left(\frac{1}{2} - 1\right) a_0'}{1 \times 2} = -\frac{a_0'}{2}.\right)$ The coefficients a_0' and a_1 are

determined from the normalization conditions

$$\int |\psi_{200}|^2 d\tau = \int |\psi_1^0|^2 d\tau = (a_0')^2 4\pi \int_0^\infty e^{-r/a}\left(1-\frac{r}{2a}\right)^2 r^2\,dr = 1$$

$$\int |\psi_{210}|^2 d\tau = \int |\psi_2^0|^2 d\tau = a_1^2 2\pi \int_0^\infty e^{-r/a}\frac{r^4\,dr}{a^2} \times \int_0^\pi \cos^2\theta \sin\theta\,d\theta = 1$$

Since $\int_0^\infty r^n e^{-r/a}\,dr = a^{n+1}n!$, we get

$$R_{20} = \frac{1}{\sqrt{8\pi a^3}} e^{-r/2a}\left(1-\frac{r}{2a}\right), \quad R_{21} = \frac{e^{-r/2a}}{\sqrt{8\pi a^3}}\frac{r}{2a}$$

Substituting these into I_r, we can write

$$I_r = \frac{1}{16\pi a^4}\int_0^\infty e^{-r/a}\left(1-\frac{r}{2a}\right) r^4\,dr$$

$$= \frac{1}{16\pi a^4}\left(4!\,a^5 - \frac{1}{2a}5!\,a^6\right) = -\frac{9a}{4\pi}$$

and, hence,

$$\langle 1 \mid W \mid 2\rangle = \langle 2 \mid W \mid 1\rangle = 3e \mid \mathbf{E} \mid a$$

Equation (2) yields

$E' = E_2^0 + 3e \mid \mathbf{E} \mid a$, $E'' = E_2^0 - 3e \mid \mathbf{E} \mid a$, $E''' = E'''' = E_2^0$

The corresponding functions are

$$\psi' = c_1'(\psi_1^0 + \psi_2^0), \qquad \psi'' = c_1''(\psi_1^0 - \psi_2^0)$$

$$\psi''' = c_3'''\psi_3^0 + c_4'''\psi_4^0, \quad \psi'''' = c_3''''\psi_3^0 + c_4''''\psi_4^0$$

From the normalization condition we get

$$c_1' = c_1'' = \frac{1}{\sqrt{2}}$$

The splitting of the energy levels of the hydrogen atom in an electric field proves to be proportional to the magnitude of the applied field.

120. In the absence of the field the Schrödinger equation for the unperturbed problem

$$\left[-\frac{\hbar^2}{2\mu}\Delta+V(r)\right]\psi=\hat{H}^0\psi=E_{nl}\psi$$

has degenerate eigenvalues: for every level E_{nl} there are $2l+1$ eigenfunctions ψ_{nlm} that differ in m while retaining the same n and l. Consequently, we must look for the wave function of the perturbed problem in the form

$$\psi=\sum_{m=-l}^{l} c_m\psi_{nlm}$$

For an atom in the external electric field $\mathbf{E}$ the equation is

$$(\hat{H}_0-e\,|\,\mathbf{E}\,|\,z)\,\psi=E\psi$$

and the term $-e\,|\,\mathbf{E}\,|\,z$ (the field is directed along the z-axis) plays the role of perturbation $\hat{W}$. We get the energy levels by making the determinant zero:

$$|\,(E_{nl}-E)\,\delta_{mm'}+\langle m\,|\,W\,|\,m'\rangle\,|=0$$

Obviously,

$$\langle m\,|\,W\,|\,m'\rangle=-e\,|\,\mathbf{E}\,|\int\psi^*_{nlm}z\psi_{nlm'}\,d\tau=-e\,|\,\mathbf{E}\,|\,I_rI_\theta I_\varphi$$

For $m\neq m'$,

$$I_\varphi=\int_0^{2\pi}e^{-im\varphi}e^{im'\varphi}\,d\varphi=0$$

For $m=m'$,

$$I_\theta=\int_0^{\pi}P_{lm}(\cos\theta)\cos\theta P_{lm}(\cos\theta)\sin\theta\,d\theta$$

$$=\int_{-1}^{1}|\,P_{lm}(x)\,|^2\,x\,dx=0$$

because the integrand is odd.

Hence, the equation for E reduces to

$$(E_{nl}-E)^{2l+1}=0,\ \text{i.e.}\ E=E_{nl}$$

There is no splitting (in this approximation) of the energy levels in the external electric field.

121. As in Problem 60, we denote the ordinary coordinates and energy as $\mathbf{r}'$ and E', and the dimensionless as $\mathbf{r} = \frac{\mathbf{r}'}{a}$ $\left(a = \frac{\hbar^2}{\mu e^2}\right)$ and $\varepsilon = \frac{E'}{e^2/a}$. Now we can write the Schrödinger equation for a hydrogen atom placed in an electric field $\mathbf{E}$ that is parallel to the z-axis:

$$-\frac{1}{2}\Delta\psi - \frac{1}{r}\psi - gz\psi = \varepsilon\psi$$

Here the perturbation is $-e\,|\,\mathbf{E}\,|\,z' = -gz$, where $g = \frac{|\mathbf{E}|a^2}{e}$ is the dimensionless analog of the external electric field.

In parabolic coordinates $u = r + z$, $v = r - z$, φ (see Appendix 3) this equation takes the form

$$\frac{\partial}{\partial u}\left(u\frac{\partial\psi}{\partial u}\right) + \frac{\partial}{\partial v}\left(v\frac{\partial\psi}{\partial v}\right) + \frac{1}{4}\left(\frac{1}{u} + \frac{1}{v}\right)\frac{\partial^2\psi}{\partial\varphi^2}$$
$$+\left[1 + \varepsilon\,\frac{u+v}{2} + \frac{g}{4}(u^2 - v^2)\right]\psi = 0$$

which permits separation of variables:

$$\psi = U(u)\,V(v)\,e^{im\varphi} \quad (m = 0, \pm 1, \pm 2, \ldots)$$

Here $U(u)$ and $V(v)$ satisfy the equations

$$\frac{d}{du}\left(u\frac{dU}{du}\right) + \left[\alpha_1 + \frac{\varepsilon u}{2} - \frac{m^2}{4u} + \frac{gu^2}{4}\right]U = 0 \qquad (1)$$

$$\frac{d}{dv}\left(v\frac{dV}{dv}\right) + \left[\beta_1 + \frac{\varepsilon v}{2} - \frac{m^2}{4v} - \frac{gv^2}{4}\right]V = 0 \qquad (2)$$

and $\alpha_1 + \beta_1 = 1$. If we compare these with the corresponding equations when $g = 0$, solved in Problem 60, where we found that

$$U^0_{n_1}(u) = e^{-u\sqrt{-\frac{\varepsilon}{2}}}F^{|m|}_{n_1}(u)\,u^{|m|/2}$$

$$\alpha = \sqrt{-\frac{\varepsilon}{2}}\,(2n_1 + |\,m\,| + 1)$$

(for the first equation), we can seek the solution of (1) using the perturbation theory. The eigenvalue α is non-

degenerate, so we must look for the correction $\Delta\alpha = \alpha_1 - \alpha$ in the form

$$\Delta\alpha = -\int_0^\infty U_{n_1}^{0*} \frac{gu^2}{4} U_{n_1}^0 \, du$$

Here

$$\int_0^\infty |U_{n_1}^0|^2 \, du = 1$$

The polynomial $F_{n_1}^{|m|}$ satisfies the equation

$$u \frac{d^2 F_{n_1}^{|m|}}{du^2} + \left(|m| + 1 - 2u\sqrt{-\frac{\varepsilon}{2}}\right) \frac{dF_{n_1}^{|m|}}{du} + \left[\alpha - (|m|+1)\sqrt{-\frac{\varepsilon}{2}}\right] F_{n_1}^{|m|} = 0$$

If we substitute the value of α and introduce a new variable

$$x = 2u\sqrt{-\frac{\varepsilon}{2}} \tag{3}$$

the equation becomes

$$x \frac{d^2 F_{n_1}^{|m|}}{dx^2} + [|m| + 1 - x] \frac{dF_{n_1}^{|m|}}{dx} + n_1 F_{n_1}^{|m|} = 0$$

Now, writing $F_{n_1}^{|m|} = \sum_{k=0}^{n_1} b_k x^k$, we find the recurrence relation for the coefficients b_k:

$$b_{k+1} = b_k \frac{k - n_1}{(k+1)(k+|m|+1)} \quad (k = 0, 1, \ldots) \tag{4}$$

We can verify explicitly that if we choose

$$b_0 = (|m| + 1)(|m| + 2) \ldots (|m| + n_1)$$

we have

$$F_{n_1}^{|m|} = \frac{e^x}{x^{|m|}} \frac{d^{n_1}(e^{-x} x^{|m|+n_1})}{dx^{n_1}} = (-1)^{n_1} x^{n_1} + \ldots \tag{5}$$

The other coefficients can be computed by using formula (4).

For an arbitrary function $f(x)$ we evaluate the integral

$$I=\int_0^\infty x^{|m|}e^{-x}F_{n_1}^{|m|}f(x)\,dx$$

Substituting (5) and integrating by parts, we find that

$$I=\int_0^\infty x^{|m|}e^{-x}\frac{e^x}{x^{|m|}}\frac{d^{n_1}(e^{-x}\cdot x^{n_1+|m|})}{dx^{n_1}}f\,dx$$

$$=(-1)^{n_1}\int_0^\infty e^{-x}x^{n_1+|m|}\frac{d^{n_1}f}{dx^{n_1}}dx$$

Using this result, we calculate the normalization integral and the matrix element. We introduce the variable (3) and write

$$U_{n_1}=C_{n_1m}e^{-x/2}F_{n_1}^{|m|}(x)\,x^{|m|/2}$$

Then

$$\int_0^\infty |U_{n_1}|^2\,du=\frac{C_{n_1m}^2}{\sqrt{-2\varepsilon}}\int_0^\infty e^{-x}x^{|m|}F_{n_1}^{|m|}F_{n_1}^{|m|}\,dx$$

$$=\frac{C_{n_1m}^2}{\sqrt{-2\varepsilon}}(-1)^{n_1}\int_0^\infty e^{-x}x^{n_1+|m|}\frac{d^{n_1}F_{n_1}^{|m|}}{dx^{n_1}}dx$$

and since $\dfrac{d^{n_1}F_{n_1}^{|m|}}{dx^{n_1}}=(-1)^{n_1}n_1!$, we have

$$C_{n_1m}^2=\frac{\sqrt{-2\varepsilon}}{n_1!\,(n_1+|m|)!}$$

In the same way

$$\Delta\alpha=-\frac{g}{4}\frac{C_{n_1m}^2}{(\sqrt{-2\varepsilon})^3}\int_0^\infty e^{-x}x^{|m|}F_{n_1}^{|m|}(F_{n_1}^{|m|}\cdot x^2)\,dx$$

$$=-\frac{g}{4}\frac{C_{n_1m}^2}{(\sqrt{-2\varepsilon})^3}(-1)^{n_1}\int_0^\infty e^{-x}x^{n_1+|m|}\frac{d^{n_1}}{dx^{n_1}}(F_{n_1}^{|m|}\cdot x^2)\,dx$$

Since

$$\frac{d^{n_1}(F_{n_1}^{|m|}x^2)}{dx^{n_1}}=b_{n_1}\frac{(n_1+2)!}{2!}x^2+b_{n_1-1}\frac{(n_1+1)!}{1!}x+b_{n_1-2}n_1!$$

and $b_{n_1} = (-1)^{n_1}$ and since formula (4) yields

$$b_{n_1-1} = \frac{n_1(n_1+|m|)}{-1} b_{n_1}, \quad b_{n_1-2} = \frac{(n_1-1)(n_1-1+|m|)}{-2} b_{n_1-1}$$

we get

$$\Delta\alpha = \frac{g}{8\varepsilon n_1!\,(n_1+|m|)!} \int\limits_0^\infty e^{-x} \left[\frac{(n_1+2)!}{2!} x^{n_1+|m|+2} \right.$$
$$- \frac{n_1(n_1+|m|)(n_1+1)!}{1} x^{n_1+|m|+1}$$
$$\left. + \frac{n_1(n_1-1)(n_1+|m|)(n_1+|m|-1)}{2!} n_1!\, x^{n_1+|m|} \right] dx$$

After simplifying we get

$$\Delta\alpha = \frac{g}{8\varepsilon} [6n_1^2 + 6n_1(|m|+1) + (|m|+1)(|m|+2)]$$

If we turn to Eq. (2), in which the term $-\frac{gv^2}{4}$ plays the role of perturbation and the eigenvalue is $-\beta_1$, we get

$$\Delta\beta = \beta_1 - \beta = \int\limits_0^\infty |V_{n_2}^{|m|}(v)|^2 \frac{gv^2}{2} dv$$
$$= -\frac{g}{8\varepsilon} [6n_2^2 + 6n_2(|m|+1) + (|m|+1)(|m|+2)]$$

(since the unperturbed equations (1) and (2) are identical). The energy ε is determined from the condition

$$1 = \alpha + \Delta\alpha + \beta + \Delta\beta = \sqrt{-\frac{\varepsilon}{2}}\, 2(n_1+n_2+|m|+1)$$
$$+ \frac{g}{8\varepsilon} [6(n_1^2 - n_2^2) + 6(|m|+1)(n_1-n_2)]$$

Since $n_1 + n_2 + |m| + 1 = n$ and in the correction term proportional to g we can replace ε by its value $\varepsilon_0 = -\frac{1}{2n^2}$ from the unperturbed problem, the final condition takes the form

$$\sqrt{-2\varepsilon}\, n \left[1 - \frac{3}{2} g n^3 (n_1 - n_2)\right] = 1$$

whence

$$\varepsilon = -\frac{1}{2n^2} - \frac{3}{2} gn (n_1 - n_2)$$

Since for a given n the numbers n_1 and n_2 can assume values from 0 to $n - 1$, the maximal value of $n_1 - n_2$ is equal to $n - 1$, and the minimal value is equal to $-(n - 1)$. Hence, the correction to ε_0 can assume $2n - 1$ values.

Thus, an energy level in the hydrogen atom under the influence of a field splits into $2n - 1$ levels, and the magnitude of the splitting is proportional to the field ($g \propto |\mathbf{E}|$).

122. We determine the electric field intensity inside the nucleus using the Gauss theorem, and bearing in mind that the potential must be continuous at $r = r_0$, we find that, for $r < r_0$, the potential energy of the electron is

$$U(r) = -\frac{3}{2} \frac{Ze^2}{r_0} \left(1 - \frac{1}{3} \frac{r^2}{r_0^2}\right)$$

i.e. the perturbation is

$$V(r) = \frac{Ze^2}{r} - U(r) \quad \text{for} \quad r \leqslant r_0$$

$$= 0 \qquad\qquad \text{for} \quad r > r_0$$

The correction to the lowest level ($n = 1$) is calculated using $\psi_0 = \psi_{100} = \sqrt{\frac{Z^3}{\pi a^3}} e^{-Zr/a}$ and is equal to V_{00}. Since $r_0 \approx 10^{-12}$ cm and $a \approx 10^{-8}$ cm, when we evaluate the integral, we can substitute unity for the exponential function and get

$$E_0^{(1)} = \frac{2}{5} \frac{Z^4 e^2}{a} \left(\frac{r_0}{a}\right)^2 = -\frac{4}{5} E_0^{(0)} \left(\frac{r_0}{a}\right)^2 Z^2$$

Even for $Z \approx 100$, the ratio $\frac{E_0^{(1)}}{E_0^{(0)}} \sim 10^{-4}$.

For the 4-fold degenerate level ($n = 2$) we find the corrections from the corresponding determinant. They are V_{11} and $V_{22} = V_{33} = V_{44}$. (The calculations are done using the functions ψ_i from Problem 56. None but the diagonal matrix elements of $V(r)$ are nonzero.) If we neglect terms of the order of r_0/a compared with unity, the corrections prove

equal since

$$E_1^{(1)} = \frac{1}{20}\frac{Z^4e^2}{a}\left(\frac{r_0}{a}\right)^2 \qquad (l=0)$$

$$E_2^{(1)} = \frac{1}{1120}\frac{Z^2e^2}{a}\left(\frac{Zr_0}{a}\right)^4 \qquad (l=1)$$

Hence, the level with $n = 2$ splits into two sublevels, the energy depending on l as well as on n.

123. The functions Ψ_0 and Ψ_1 satisfy the equation

$$\hat{H}\Psi_n = i\hbar\frac{\partial\Psi_n}{\partial t}$$

and

$$\Psi_n = u_n(r)\,e^{-\frac{i}{\hbar}E_n t}, \quad \int \Psi_n^*\Psi_m\,d\tau = \delta_{nm} \quad (n,\ m=0,\ 1)$$

After the perturbation $\hat{W}$ is switched on, the Schrödinger equation

$$(\hat{H}+\hat{W})\,\Psi = i\hbar\frac{\partial\Psi}{\partial t}$$

is to be solved in terms of the stationary states:

$$\Psi = a_0(t)\,\Psi_0 + a_1(t)\,\Psi_1$$

where $|a_0|^2 + |a_1|^2 = 1$, and the initial values of the coefficients are $a_0(0) = 1$ and $a_1(0) = 0$. Substituting Ψ into this equation, we find by the usual method that

$$i\hbar\dot{a}_0 = a_0W_{00} + a_1W_{10}e^{\frac{i}{\hbar}(E_0-E_1)t}$$

$$i\hbar\dot{a}_1 = a_0W_{01}e^{\frac{i}{\hbar}(E_1-E_0)t} + a_1W_{11}$$

where $W_{ik} = \int u_i^*\hat{W}u_k\,d\tau$ are the matrix elements of $\hat{W}$. If we introduce $\alpha_0(t) = a_0(t)$ and $\alpha_1(t) = e^{i(E_0-E_1)t/\hbar}a_1(t)$, the equations for α_i

$$i\hbar\dot{\alpha}_0 = \alpha_0W_{00} + \alpha_1W_{10}$$

$$i\hbar\dot{\alpha}_1 = \alpha_0W_{01} + \alpha_1[W_{11} + E_1 - E_0]$$

are homogeneous linear equations with constant coefficients, and α_i can be sought in the form $\alpha_0 = Ae^{-i\Omega t}$ and $\alpha_1 =$

$= Be^{-i\Omega t}$. Coefficients A and B are found from the equations

$$(W_{00} - \hbar\Omega) A + W_{10} B = 0$$
$$W_{01} A + [W_{11} + E_1 - E_0 - \hbar\Omega] B = 0$$

The determinant of these two equations vanishes for the two frequencies Ω_{12}:

$$\hbar\Omega_{12} = W_{00} + \frac{\gamma}{2} \pm \sqrt{|W_{10}|^2 + \frac{\gamma^2}{4}}$$

with

$$\gamma = W_{11} - W_{00} + E_1 - E_0$$

Hence,

$$\alpha_0 = A_1 e^{-i\Omega_1 t} + A_2 e^{-i\Omega_2 t}, \quad \alpha_1 = B_1 e^{-i\Omega_1 t} + B_2 e^{-i\Omega_2 t}$$

where

$$B_i = \frac{\hbar\Omega_i - W_{00}}{W_{10}} A_i \quad (i = 1, 2) \tag{1}$$

The initial conditions permit evaluation of the constants of integration:

$$A_1 + A_2 = 1, \quad B_1 + B_2 = 0 \tag{2}$$

and, hence,

$$\alpha_0 = A_1 (e^{-i\Omega_1 t} - e^{-i\Omega_2 t}) + e^{-i\Omega_2 t}$$
$$\alpha_1 = B_1 (e^{-i\Omega_1 t} - e^{-i\Omega_2 t})$$

If we substitute (1) into (2), we get

$$A_1 = \frac{W_{00} - \hbar\Omega_2}{\hbar(\Omega_1 - \Omega_2)}$$
$$B_1 = -\frac{(\hbar\Omega_1 - W_{00})(\hbar\Omega_2 - W_{00})}{\hbar(\Omega_1 - \Omega_2) W_{10}} = \frac{W_{10}}{\hbar(\Omega_1 - \Omega_2)}$$

Q.E.D.

Let us now estimate the behaviour of $|a_i|^2 = |\alpha_i(t)|^2$ in time. Calculating the modulus in the usual way, we find that

$$|\alpha_0|^2 = 1 + 4(A_1^2 - A_1)\sin^2 \sigma t$$

and

$$|\alpha_1|^2 = 4B_1^2 \sin^2 \sigma t$$

where

$$\sigma = \frac{\Omega_1 - \Omega_2}{2} = \frac{\sqrt{|W_{10}|^2 + \frac{\gamma^2}{4}}}{\hbar}$$

Using the expressions for Ω_1 and Ω_2, we can show that

$$4(A_1^2-A_1)=(2A_1-1)^2-1=\left[\frac{2W_{00}-\hbar(\Omega_1+\Omega_2)}{\hbar(\Omega_1-\Omega_2)}\right]^2-1$$

$$=\frac{\gamma^2}{4\sigma^2}-1=-\frac{|W_{10}|^2}{\sigma^2}$$

since $4\sigma^2-\gamma^2=4|W_{10}|^2$ and, hence,

$$|\alpha_0|^2=|a_0|^2=1-\frac{|W_{10}|^2}{\sigma^2}\sin^2\sigma t$$

$$|\alpha_1|^2=|a_1|^2=\frac{|W_{10}|^2}{\sigma^2}\sin^2\sigma t$$

The perturbation results in the system oscillating between the two states Ψ_0 and Ψ_1 with a frequency

$$\sigma=\frac{1}{\hbar}\sqrt{|W_{10}|^2+\frac{\gamma^2}{4}}.$$

124. The probability of transition is determined by formula (III-34). To compute $C_k(t)$ we bear in mind that W_{kn} = constant for $0\leqslant t\leqslant\tau$ and is equal to zero for all other t. Hence,

$$C_k(t)=\frac{W_{kn}}{i\hbar}\int\limits_0^\tau e^{i\omega_{kn}t}\,dt=2\,\frac{W_{kn}}{i\hbar}\,e^{i\omega_{kn}\tau/2}\,\frac{\sin\frac{\omega_{kn}\tau}{2}}{\omega_{kn}}$$

and

$$P_{k\to n}=\frac{4}{\hbar^2}|W_{kn}|^2\frac{\sin^2\left(\frac{E_k-E_n}{2\hbar}\tau\right)}{\left(\frac{E_k-E_n}{\hbar}\right)^2}=\frac{4}{\hbar^2}|W_{kn}|^2F(E_k-E_n)$$

The function $F(E_k-E_n)$ has a maximum at $E_n=E_k$, and if τ is sufficiently large, we can express F using the delta function. Let us consider the integral

$$J=\int\limits_{-\infty}^{\infty}f(E_k)\,F(E_k-E_n)\,dE_k$$

We substitute $\frac{E_k-E_n}{2\hbar}\tau=z$ and $dE_k=2\hbar\frac{dz}{\tau}$, and we find that

$$J=\int\limits_{-\infty}^{\infty}f\left(E_n+\frac{2\hbar z}{\tau}\right)\frac{\sin^2 z}{z^2}\,\frac{\hbar}{2}\,dz\times\tau$$

Now if we direct τ to infinity, we get

$$J=\tau\pi\frac{\hbar}{2}f(E_n), \text{ i.e. } F(E_k-E_n)=\tau\pi\frac{\hbar}{2}\delta(E_k-E_n)$$

and, hence,

$$P_{k\to n}=\frac{2\pi}{\hbar}\tau\,|W_{kn}|^2\delta(E_k-E_n).$$

125. The probability of transition from the state ψ_1 into ψ_p under the influence of the perturbation $\hat{W}(\mathbf{r}, t)$ is determined in the first approximation by the coefficient

$$c_p(T)=\frac{1}{i\hbar}\int_0^T W_{p1}(t)\,e^{i\frac{E_p-E_1}{\hbar}t}\,dt$$

where $W_{p1}=\int\psi_p^*\hat{W}(\mathbf{r}, t)\,\psi_1\,d\tau$, and E_1 and E_p are the energies of the corresponding states. For the initial state we must write the wave function of the electron in the Coulomb field of the nucleus Ze with $n=1$, $l=0$, $m=0$. This function is $\psi_1=\psi_{100}=Ce^{-rZ/a}$, where $a=\frac{\hbar^2}{\mu e^2}$. From the normalization condition we have $C=\sqrt{\frac{Z^3}{\pi a^3}}$. In the final state the electron is described by a plane wave $\psi_p=\frac{1}{\sqrt{V}}e^{i\mathbf{pr}/\hbar}$, where V is the volume of the region where the atom is located. To determine the number of states with a given momentum we impose, as usual, the requirement of periodicity, $\psi_p(x+L)=\psi_p(x)$ (where $L^3=V$), and we get the allowed values of momentum:

$$p_n=\frac{hn}{L}$$

It follows from this that the number of states with momenta in the interval $[\mathbf{p}, \mathbf{p}+d\mathbf{p}]$ is equal to $\rho\,dp_xdp_ydp_z=2\frac{dp_xdp_ydp_z}{h^3}V$ (the factor 2 indicates the number of possible spin projections). Now if we use spherical coordinates in the momentum space, we can write the number of states with the energy in the interval $[E_p, E_p+dE_p]$ $\left(\text{where } E_p=\frac{p^2}{2\mu}\right)$ and with the direction of the momentum

in the angle $d\Omega$ as

$$\rho(E_p)\,dE_p\,d\Omega = \frac{2mp}{h^3}\,d\Omega\,dE_p$$

For this problem the vector potential is given as

$$A_x = A\cos(\omega t - \mathbf{kr}), \qquad A_y = A_z = 0 \ (\mathbf{k} \parallel OZ)$$

Hence,

$$\hat{W} = -\frac{e}{\mu}(\mathbf{A}\cdot\hat{\mathbf{p}}) = -\frac{eA}{\mu}\cos(\omega t - \mathbf{kr}) \times \hat{p}_x$$

Since $W_{p1} = \int \psi_p^* \hat{W} \psi_1\,d\tau = W_{1p}^* = \left[\int \psi_1^* \hat{W}\psi_p\,d\tau\right]^*$ and $\hat{p}_x\psi_p = p_x\psi_p$, it follows that the matrix element W_{p1} can be written in the form

$$W_{p1} = -\frac{eAp_x}{\mu}\int \psi_p^* \cos(\omega t - \mathbf{kr})\,\psi_1\,d\tau$$

Representing $\cos(\omega t - \mathbf{kr})$ in the form

$$\cos(\omega t - \mathbf{kr}) = \frac{1}{2}\left[e^{i(\omega t - \mathbf{kr})} + e^{-i(\omega t - \mathbf{kr})}\right]$$

we can write

$$c_p(T) = -\frac{ep_xA}{2i\hbar\mu}\left[\int \psi_p^* e^{-i\mathbf{kr}}\psi_1\,d\tau \int_0^T e^{\frac{i}{\hbar}(E_p - E_1 + \hbar\omega)t}\,dt + \int \psi_p^* e^{i\mathbf{kr}}\psi_1\,d\tau \int_0^T e^{\frac{i}{\hbar}(E_p - E_1 - \hbar\omega)t}\,dt\right]$$

Since

$$\int_0^T e^{\frac{i}{\hbar}(E_p - E_1 \pm \hbar\omega)t}\,dt = \frac{e^{\frac{i}{\hbar}(E_p - E_1 \pm \hbar\omega)T} - 1}{(E_p - E_1 \pm \hbar\omega)\frac{i}{\hbar}},$$

of the two terms in $c_p(T)$, the one in which the denominator contains $E_p - E_1 - \hbar\omega \approx 0$ plays the main role in the photoelectric effect ($E_p > E_1$). We neglect the second term and substitute the plane wave and the Coulomb wave function for ψ_p and ψ_1, respectively, and we get

$$c_p(T) = -\frac{eAp_x}{2\hbar\mu}\sqrt{\frac{Z^3}{\pi a^3 V}}\,\frac{e^{\frac{i}{\hbar}(E_p - E_1 - \hbar\omega)T} - 1}{[E_p - E_1 - \hbar\omega]\frac{i}{\hbar}}\,I$$

where

$$I=\int e^{-Zr/a}e^{i\left(\mathbf{kr}-\frac{\mathbf{pr}}{\hbar}\right)}\,d\tau=\int_0^\infty e^{-Zr/a}r^2\,dr\int_0^\pi e^{-iqr\cos\theta}$$

$$\times\sin\theta\,d\theta\int_0^{2\pi}d\varphi$$

Here $\frac{\mathbf{p}}{\hbar}-\mathbf{k}=\mathbf{q}$, and the z-axis is directed along $\mathbf{q}$ so that $(\mathbf{q}\cdot\mathbf{r})=qr\cos\theta$. Then, since $\int_0^\infty e^{-\alpha r}r\,dr=1/\alpha^2$, we have

$$I=\frac{2\pi}{iq}\left[\int_0^\infty e^{-r\left(\frac{Z}{a}-iq\right)}r\,dr-\int_0^\infty e^{-r\left(\frac{Z}{a}+iq\right)}r\,dr\right]=\frac{8\pi\frac{Z}{a}}{\left(\frac{Z^2}{a^2}+q^2\right)^2}$$

The probability of transition into the interval ΔE_p,

$$dW=\int_{\Delta E_p}|c_p(T)|^2\rho(E_p)\,dE_p\,d\Omega$$

contains along with a smoothly varying function of E the expression

$$\frac{\left|e^{i(E_p-E_1-\hbar\omega)\frac{T}{\hbar}}-1\right|^2}{(E_p-E_1-\hbar\omega)^2\frac{1}{\hbar^2}}dE_p=\frac{\sin^2\frac{E_p-E_1-\hbar\omega}{2\hbar}T}{\left(\frac{E_p-E_1-\hbar\omega}{2\hbar}T\right)^2}\frac{dE_p}{2\hbar}2\hbar T^2$$

$$=2\pi\hbar T\delta(E_p-E_1-\hbar\omega)\,dE_p,$$

For this reason the probability of the photoelectron being emitted into the angle $d\Omega$ in unit time is

$$\frac{dW}{T}=32\pi\frac{e^2}{\mu}\frac{A^2Z^5}{h^3a^5}\frac{p_x^2p}{\left(\frac{Z^2}{a^2}+q^2\right)^4}\frac{2\pi}{\hbar}\delta(E_p-E_1-\hbar\omega)\,d\Omega$$

with $E_p=E_1+\hbar\omega=\hbar\omega-J$, where J is the energy of ionization of the atom; $E_1=-J=-\frac{\mu e^4Z^2}{2\hbar^2}$. Substituting

the explicit expression for q^2 into the denominator, we get

$$\frac{Z^2}{a^2}+q^2=\frac{Z^2}{a^2}+\frac{p^2}{\hbar^2}+k^2-\frac{2pk}{\hbar}\cos\alpha$$
$$=\frac{2\mu}{\hbar^2}\left[\frac{\mu Z^2 e^4}{2\hbar^2}+\frac{p^2}{2\mu}\right]+k^2-\frac{2pk}{\hbar}\cos\alpha$$
$$=\frac{2\mu}{\hbar^2}\hbar\omega+k^2-\frac{2pk}{\hbar}\cos\alpha\approx\frac{2\mu\omega}{\hbar}\left(1-\frac{v}{c}\cos\alpha\right)$$

if we neglect $k^2 \ll p^2/\hbar^2$.

Now if we direct vector $\mathbf{k}$ along the z-axis, we get $p_x = p\sin\alpha\cos\varphi$ and finally

$$\frac{dW}{T}=\frac{e^2}{\mu}\,\frac{A^2}{2\pi a^5\mu^4}\,\frac{Z^5}{\omega^4}\,\frac{\sin^2\alpha\cos^2\varphi}{\left(1-\frac{v}{c}\cos\alpha\right)^4}\,p^3\,d\Omega\,\delta\left(\frac{p^2}{2\mu}+J-\hbar\omega\right).$$

126. According to the conditions of the problem the equations

$$\hat{H}_i u_n(\mathbf{r}_i)\equiv-\frac{\hbar^2}{2m}\Delta_i u_n+V(\mathbf{r}_i)\,u_n=E_n u_n(\mathbf{r}_i)\qquad(i=1,\,2)$$

have been solved, and we must find the eigenfunctions and eigenvalues of the operator $\hat{H}=\hat{H}_1+\hat{H}_2+\hat{H}_{12}$ assuming that $\hat{H}_{12}=\hat{H}_{21}$. We view $\hat{H}_{12}$ as the perturbation and first solve the unperturbed problem

$$\hat{H}_0\Psi_{01}(\mathbf{r}_1,\mathbf{r}_2)=(\hat{H}_1+\hat{H}_2)\,\Psi_{01}(\mathbf{r}_1,\mathbf{r}_2)=E_0\Psi_{01}(\mathbf{r}_1,\mathbf{r}_2)$$

Obviously, we can write

$$\Psi_{01}(\mathbf{r}_1,\mathbf{r}_2)=u_r(\mathbf{r}_1)\,u_s(\mathbf{r}_2)$$

The energy level corresponding to this function is $E_0 = E_r + E_s$. But we can bring another function into conformity with this energy level, namely,

$$\Psi_{02}(\mathbf{r}_1,\mathbf{r}_2)=u_r(\mathbf{r}_1)\,u_r(\mathbf{r}_2)$$

(The particles have exchanged states. The first is now in the s state, and the second in the r state.) Since the energy level E_0 is two-fold degenerate, we look for the eigenfunction of the perturbed problem in the form

$$\Psi=a\Psi_{01}+b\Psi_{02}$$

We substitute this into the equation $(\hat{H}_1+\hat{H}_2+\hat{H}_{12}-E)\times\Psi=0$ and denote $E=E_r+E_s+\varepsilon$, and we get the

simplified equation

$$(\hat{H}_{12} - \varepsilon)(a\Psi_{01} + b\Psi_{02}) = 0$$

Next we multiply this first by Ψ_{01}^* and then by Ψ_{02}^* and integrate over the entire space. We find that we have two equations for a and b:

$$(C_{11} - \varepsilon)\, a + C_{12}b = 0 \quad (1)$$

$$C_{21}a + (C_{22} - \varepsilon)\, b = 0 \quad (2)$$

Since $\hat{H}_{12}$ is symmetric, the coefficients $C_{11} = C_{22}$ and $C_{12} = C_{21}$. Indeed, if we replace $\mathbf{r}_1$ by $\mathbf{r}_2$ and $\mathbf{r}_2$ by $\mathbf{r}_1$, this transforms Ψ_{01} into Ψ_{02} and, hence, the coefficient

$$C_{11} = \int\int \Psi_{01}^* \hat{H}_{12} \Psi_{01}\, d\tau_1\, d\tau_2$$
$$= \int\int u_r^*(\mathbf{r}_1)\, u_s^*(\mathbf{r}_2)\, \hat{H}_{12} u_r(\mathbf{r}_1)\, u_s(\mathbf{r}_2)\, d\tau_1\, d\tau_2$$

differs only in the labeling of the variables from

$$C_{22} = \int\int \Psi_{02}^* \hat{H}_{12} \Psi_{02}\, d\tau_1\, d\tau_2$$
$$= \int\int u_r^*(\mathbf{r}_2)\, u_s(\mathbf{r}_1)\, \hat{H}_{12} u_r(\mathbf{r}_2)\, u_s(\mathbf{r}_1)\, d\tau_1\, d\tau_2$$

In the same way,

$$C_{12} = \int \Psi_{01}^* \hat{H}_{12} \Psi_{02}\, d\tau_1\, d\tau_2 = C_{21} = \int \Psi_{02}^* \hat{H}_{21} \Psi_{01}\, d\tau_2\, d\tau_1$$

since $\hat{H}_{12} = \hat{H}_{21}$. We denote $C_{11} = C_{22} = K$ and $C_{12} = C_{21} = A$ and write equations (1) and (2) in the form

$$(K - \varepsilon)\, a + Ab = 0$$
$$Aa + (K - \varepsilon)\, b = 0$$

From the condition that the determinant vanishes we find that

$$K - \varepsilon = \pm A$$

(1) $\varepsilon = K + A$ and $a = b$. Hence, the level $E' = E_r + E_s + K + A$ has the corresponding function

$$\Psi' = a(\Psi_{01} + \Psi_{02}) = a\,[u_r(\mathbf{r}_1)\, u_s(\mathbf{r}_2) + u_s(\mathbf{r}_1)\, u_r(\mathbf{r}_2)]$$

(2) $\varepsilon = K - A$ and $a = -b$. The level $E'' = E_r + E_s + K - A$ has the corresponding function

$$\Psi'' = a\,(\Psi_{01} - \Psi_{02}) = a\,[u_r(\mathbf{r}_1)\,u_s(\mathbf{r}_2) - u_s(\mathbf{r}_1)\,u_r(\mathbf{r}_2)]$$

From the normalization condition we find that $a = \frac{1}{\sqrt{2}}$ since

$$\int |\Psi_{01}|^2\,d\tau_1\,d\tau_2 = \int |\Psi_{02}|^2\,d\tau_1\,d\tau_2 = 1$$

and

$$\int \Psi_{01}^*\Psi_{02}\,d\tau_1\,d\tau_2 = \left(\int u_r^*(\mathbf{r}_1)\,u_s(\mathbf{r}_1)\,d\tau_1\right)^2 = 0$$

If the particles involved are electrons, the complete function $\Phi(\mathbf{r}_1, \mathbf{r}_2, \sigma_1, \sigma_2)$, i.e. the spatial and spin functions, must be antisymmetric. Hence, the level

$$E' = E_r + E_s + K + A$$

has one function corresponding to it:

$$\Phi_a = \frac{1}{\sqrt{2}}\,(\Psi_{01} + \Psi_{02})\,\chi_a(\sigma_1, \sigma_2)$$

(There is only one antisymmetric spin two-electron function; see Problem 110.) Consequently, this is the singlet state. The second eigenvalue

$$E'' = E_r + E_s + K - A$$

corresponds to the triplet state, since we can combine with the antisymmetric spatial function $\Psi'' = \frac{1}{\sqrt{2}}(\Psi_{01} - \Psi_{02})$ three symmetric spin functions

$$\chi_c' = \alpha_1\alpha_2, \quad \chi_c'' = \beta_1\beta_2, \quad \chi_c''' = \frac{1}{\sqrt{2}}\,(\alpha_1\beta_2 + \beta_1\alpha_2).$$

127. The wave function of the system at the initial time is given in the form (see the solution to Problem 126)

$$\Psi_{t=0} = u_r(\mathbf{r}_1)\,u_s(\mathbf{r}_2) = \Psi_{01} = \Psi(0)$$

To determine its change in time, we represent $\Psi(t)$ as a result of the action of a certain operator $\hat{S}$ on $\Psi(0)$: $\Psi(t) = \hat{S}\Psi(0)$. We substitute this function into the time-depen-

dent Schrödinger equation:

$$i\hbar \frac{\partial \hat{S}}{\partial t} \Psi(0) = \hat{H}\hat{S}\Psi(0)$$

After a formal integration over t we find that

$$\hat{S} = e^{-\frac{i}{\hbar}\hat{H}t} \quad \text{and} \quad \Psi(t) = e^{-\frac{i}{\hbar}\hat{H}t}\Psi(0)$$

If we expand Ψ (0) in a complete set of eigenfunctions of $\hat{H}$, which satisfy the equation $H\psi_n = E_n\psi_n$, we get

$$\Psi(t) = e^{-\frac{i}{\hbar}\hat{H}t} \sum_n c_n\psi_n = \sum_n c_n e^{-\frac{i}{\hbar}E_n t}\psi_n$$

since the action of any operator function $f(\hat{H})$ on the eigenfunctions of $\hat{H}$ is

$$f(\hat{H})\psi_n = f(E_n)\psi_n$$

Hence, to determine $\Psi(t)$ we must first expand its initial value Ψ (0) in a complete set of eigenfunctions of $\hat{H}$, i.e. in a series in ψ_n.

In Problem 126 we found the eigenfunctions of H in the zero approximation.

The function $\Psi' = \frac{1}{\sqrt{2}}(\Psi_{01} + \Psi_{02})$ corresponding to the level $E' = E_r + E_s + K + A$ and $\Psi'' = \frac{1}{\sqrt{2}}(\Psi_{01} - \Psi_{02})$ corresponding to the level $E'' = E_r + E_s + K - A$.

From this we find that

$$\Psi_{01} = \frac{1}{\sqrt{2}}(\Psi' + \Psi'') = \Psi(0)$$

and according to what was proved earlier

$$\Psi(t) = \frac{1}{\sqrt{2}}\left(e^{-\frac{i}{\hbar}E't}\Psi' + e^{-\frac{i}{\hbar}E''t}\Psi''\right)$$

If we substitute E' and E'' and return to Ψ_{01} and Ψ_{02}, we find that

$$\Psi(t) = \frac{1}{2}e^{-\frac{i}{\hbar}(E_r+E_s+K)t}\left[(\Psi_{01}+\Psi_{02})e^{-\frac{i}{\hbar}At} + (\Psi_{01}-\Psi_{02})e^{\frac{i}{\hbar}At}\right]$$
$$= c_1(t)\Psi_{01} + c_2(t)\Psi_{02}$$

where

$$c_1(t)=\exp\left[-\frac{i}{\hbar}(E_r+E_s+K)t\right]\cos\frac{At}{\hbar}$$

$$c_2(t)=\exp\left[-\frac{i}{\hbar}(E_r+E_s+K)t\right]\sin\frac{At}{\hbar}$$

i.e. the probabilities of finding the system in the Ψ_{01} or Ψ_{02} states is, respectively,

$$W_{01}=|c_1|^2=\cos^2\frac{At}{\hbar},\quad W_{02}=|c_2|^2=\sin^2\frac{At}{\hbar}$$

The time needed for the particles to exchange states, i.e. for $\Psi_{01}=u_r(\mathbf{r}_1)\,u_s(\mathbf{r}_2)$ to transform into $\Psi_{02}=u_r(\mathbf{r}_2)\times\times u_s(\mathbf{r}_1)$, is determined by the fact that $c_1(\tau)=0$, i.e. $A\tau/\hbar=\pi/2$, whence,

$$\tau=\frac{\pi\hbar}{2A}.$$

128. The parameters A and α of the trial function $\varphi = A(1+\alpha r)\,e^{-\alpha r}$ are chosen such that the expectation value of the Hamiltonian

$$\langle H\rangle=\int\varphi^*\hat{H}\varphi\,d\tau$$

calculated via this function will become a minimum provided the set of trial functions are normalized. This last requirement gives

$$\int|\varphi|^2\,d\tau=4\pi A^2\int_0^\infty(1+\alpha r)^2\,e^{-2\alpha r}r^2\,dr$$

$$=4\pi A^2\left[\frac{2!}{(2\alpha)^3}+2\alpha\frac{3!}{(2\alpha)^4}+\frac{4!\alpha^2}{(2\alpha)^5}\right]=\frac{7\pi A^2}{\alpha^3}=1$$

i.e.

$$A^2=\frac{\alpha^3}{7\pi}$$

The Hamiltonian of a three-dimensional isotropic oscillator

$$\hat{H}=\hat{T}+\hat{V}=-\frac{\hbar^2}{2m}\Delta+\frac{m\omega^2}{2}r^2$$

Since $\hat{\mathbf{p}}$ is hermitian, we can easily transform $\langle T\rangle$:

$$\langle T\rangle=\frac{1}{2m}\int\varphi^*\hat{p}^2\varphi\,d\tau=\frac{1}{2m}\int|\hat{\mathbf{p}}\varphi|^2\,d\tau=\frac{\hbar^2}{2m}\int|\operatorname{grad}\varphi|^2\,d\tau$$

and since φ does not depend on the angles, we get

$$|\operatorname{grad}\varphi| = \left|\frac{d\varphi}{dr}\right| = A\alpha^2 re^{-\alpha r}$$

and

$$\langle T\rangle = \frac{\hbar^2}{2m} A^2\alpha^4 4\pi \int_0^\infty r^4 e^{-2\alpha r}\,dr = \frac{3}{14}\frac{\hbar^2\alpha^2}{m}$$

Calculation of $\langle V\rangle$ gives us

$$\langle V\rangle = \int \varphi^* \frac{m\omega^2}{2} r^2\varphi\,d\tau = 4\pi A^2 \frac{m\omega^2}{2}\int_0^\infty (1+\alpha r)^2 e^{-2\alpha r} r^4\,dr$$

$$= \frac{81}{28}\frac{m\omega^2}{\alpha^2}$$

and, hence,

$$\langle H\rangle \equiv \overline{H}(\alpha) = \frac{3}{14}\frac{\hbar^2\alpha^2}{m} + \frac{81}{28}\frac{m\omega^2}{\alpha^2}$$

The normalization condition is considered explicitly, and we must find the minimum of $\overline{H}(\alpha)$. Obviously, α_0 is determined from the equation

$$\frac{\partial}{\partial\alpha}\left(\frac{3}{14}\frac{\hbar^2\alpha^2}{m} + \frac{81}{28}\frac{m\omega^2}{\alpha^2}\right)\Bigg|_{\alpha_0} = 0$$

and is equal to

$$\alpha_0^2 = 3\sqrt{\frac{3}{2}}\,\frac{m\omega}{\hbar}$$

and thus

$$\overline{H}(\alpha_0) = \langle H\rangle_{\min} = \frac{9}{7}\sqrt{\frac{3}{2}}\,\hbar\omega \approx 1.575\,\hbar\omega$$

which is only 5 per cent more than the exact value of the lowest energy level for this system, $E_0 = \frac{3}{2}\hbar\omega$.

129. The normalization condition for the function $\varphi = Be^{-\alpha r/2a}$ is

$$\int |\varphi|^2\,d\tau = 4\pi B^2\int_0^\infty e^{-\alpha r/a} r^2\,dr = 4\pi B^2\frac{2!a^3}{\alpha^3} = 1$$

and so

$$B^2 = \frac{\alpha^3}{8\pi a^3}$$

As in Problem 128, we compose $\frac{d\varphi}{dr} = -B\frac{\alpha}{2a}e^{-\alpha r/2a}$ and calculate

$$\langle T\rangle = \int \left|\frac{d\varphi}{dr}\right|^2 d\tau \times \frac{\hbar^2}{2m} = \frac{\hbar^2}{2m}B^2\frac{\alpha^2}{4a^2}4\pi\int_0^\infty e^{-\alpha r/a}r^2\,dr = \frac{\hbar^2\alpha^2}{8ma^2}$$

and

$$\langle V\rangle = -A\int \varphi^* e^{-r/a}\varphi\,d\tau = -4\pi AB^2\int_0^\infty e^{-(\alpha+1)\frac{r}{a}}r^2\,dr$$

$$= -A\frac{\alpha^3}{(1+\alpha)^3}$$

From the condition of the minimum of the function

$$\langle H\rangle = \frac{\hbar^2\alpha^2}{8ma^2} - \frac{A\alpha^3}{(1+\alpha)^3}$$

we get

$$\frac{\hbar^2\alpha_0}{4ma^2} - \frac{3A\alpha_0^2}{(1+\alpha_0)^4} = 0$$

Only numerical methods are suitable to solve this equation of the fourth power in α. For this we must have the concrete values of the parameters of the problem. The ground state of the deuteron will then be equal to $\langle H\rangle(\alpha_0)$.

130. For the hydrogen atom, $\hat{H} = -\frac{\hbar^2}{2m}\Delta - \frac{e^2}{r}$. As in Problem 128, we compute the sought quantities using both functions, and we get

$$\langle H\rangle_{\min}[\varphi_1] \approx -\frac{me^4}{2.07\hbar^2}$$

$$\langle H\rangle_{\min}[\varphi_2] \approx -\frac{me^4}{2.3\hbar^2}$$

Clearly, φ_1 is a better trial function than φ_2 since $\bar{H}_{\min}[\varphi_1] < \bar{H}_{\min}[\varphi_2]$.

131. Let us represent the Hamiltonian of the problem of two electrons near the nucleus

$$\hat{H} = -\frac{\hbar^2}{2m}\Delta_1 - \frac{\hbar^2}{2m}\Delta_2 - \frac{2e^2}{r_1} - \frac{2e^2}{r_2} + \frac{e^2}{r_{12}}$$

in the form $\hat{H} = \hat{H}_1 + \hat{H}_2 + \hat{H}_{12}$, where

$$\hat{H}_1 = -\frac{\hbar^2}{2m}\Delta_1 - \frac{(2-s)\,e^2}{r_1}$$

$$\hat{H}_2 = -\frac{\hbar^2}{2m}\Delta_2 - \frac{(2-s)\,e^2}{r_2}$$

$$\hat{H}_{12} = \frac{e^2}{r_{12}} - \frac{se^2}{r_1} - \frac{se^2}{r_2}$$

The equation $\bar{H}_1 u\,(r_1) = E_0 u\,(r_1)$ is the Schrödinger equation for a hydrogen atom but with a charge of the nucleus $(2-s)\,e$ (we shall determine s in such a way that E is minimal). Consequently,

$$E_0 = -\frac{me^4}{2\hbar^2}(2-s)^2;$$

and the corresponding normalized eigenfunction is

$$u = \frac{\sqrt{\gamma^3}}{\sqrt{\pi}}e^{-\gamma r_1}, \quad \text{where} \quad \gamma = \frac{me^2}{\hbar^2}(2-s)$$

The ground state of helium in the zero approximation (the unperturbed problem) is described by the function

$$\Psi\,(\mathbf{r}_1, \mathbf{r}_2) = u\,(\mathbf{r}_1)\,u\,(\mathbf{r}_2)$$

We must calculate the energy correction $\varepsilon = E - 2E_0$ as

$$\varepsilon = \int \Psi^* \hat{H}_{12}\Psi\,d\tau_1\,d\tau_2$$

$$= \frac{e^2\gamma^6}{\pi^2}\int d\tau_1 \int d\tau_2 e^{-2\gamma(r_1+r_2)}\left(\frac{1}{r_{12}} - \frac{s}{r_1} - \frac{s}{r_2}\right) \qquad (1)$$

It is easiest to integrate using elliptical coordinates, placing the nucleus and the first electron in the focuses and denoting

$$r_1 = 2c, \; r_2 = c\,(\xi - \eta), \; r_{12} = c\,(\xi + \eta)$$

$$d\tau_2 = c^3\,(\xi^2 - \eta^2)\,d\xi d\eta d\varphi$$

The coordinates ξ, η, φ change within the limits

$$1 \leqslant \xi \leqslant \infty, \qquad -1 \leqslant \eta \leqslant 1, \qquad 0 \leqslant \varphi \leqslant 2\pi$$

Then

$$\varepsilon = \frac{e^2\gamma^6}{\pi^2} \int d\tau_1 e^{-2\gamma r_1} \times 2\pi c^2 \int_1^\infty d\xi \int_{-1}^{1} d\eta\, (\xi^2 - \eta^2)$$

$$\times\, e^{-2\gamma c(\xi-\eta)} \left(\frac{1}{\xi+\eta} - \frac{s}{2} - \frac{s}{\xi-\eta} \right)$$

$$= \frac{e^2\gamma^6}{\pi^2} \int d\tau_1 e^{-2\gamma r_1} 2\pi c^2 I(r_1) \qquad (2)$$

We evaluate the following integrals over ξ and η:

$$I_0 = \int_1^\infty e^{-2\gamma c\xi}\, d\xi = \frac{e^{-2\gamma c}}{2\gamma c}, \quad I_1 = \int_1^\infty e^{-2\gamma c\xi}\, \xi\, d\xi$$

$$= \frac{e^{-2\gamma c}}{2\gamma c} \left(1 + \frac{1}{2\gamma c} \right)$$

$$I_2 = \int_1^\infty e^{-2\gamma c\xi}\, \xi^2\, d\xi = \frac{e^{-2\gamma c}}{2\gamma c} \left[1 + \frac{2}{2\gamma c} + \frac{2}{(2\gamma c)^2} \right]$$

$$A_0 = \int_{-1}^{1} e^{2\gamma c\eta}\, d\eta = \frac{e^{2\gamma c} - e^{-2\gamma c}}{2\gamma c}$$

$$A_1 = \int_{-1}^{1} e^{2\gamma c\eta}\, \eta\, d\eta = \frac{e^{2\gamma c}}{2\gamma c} \left(1 - \frac{1}{2\gamma c} \right) + \frac{e^{-2\gamma c}}{2\gamma c} \left(1 + \frac{1}{2\gamma c} \right)$$

$$A_2 = \int_{-1}^{1} e^{2\gamma c\eta}\, \eta^2\, d\eta = \frac{e^{2\gamma c}}{2\gamma c} \left[1 - \frac{2}{2\gamma c} + \frac{2}{(2\gamma c)^2} \right]$$

$$- \frac{e^{-2\gamma c}}{2\gamma c} \left[1 + \frac{2}{2\gamma c} + \frac{2}{(2\gamma c)^2} \right]$$

These integrals enter into $I(r_1)$ in the following way:

$$I(r_1) = \frac{s}{2}(I_0 A_2 - I_2 A_0) - (s+1) I_0 A_1 + (1-s) I_1 A_0$$

After substituting the computed I_k and A_k, we get

$$I(r_1) = \frac{2}{(2\gamma c)^3} [1 - s - 2\gamma c s - e^{-4\gamma c}(2\gamma c + 1)]$$

$$= \frac{2}{\gamma^3 r_1^3} [1 - s - \gamma s r_1 - e^{-2\gamma r_1}(1 + \gamma r_1)]$$

and, introducing it into (2), we integrate over r_1:

$$\varepsilon=\frac{e^2\gamma^6}{\pi^2}\int_0^\infty 4\pi r_1^2\,dr_1 e^{-2\gamma r_1}\times 2\pi\left(\frac{r_1}{2}\right)^2 I(r_1)$$

$$=4e^2\gamma^3\int_0^\infty [(1-s)\,r_1e^{-2\gamma r_1}-s\gamma r_1^2e^{-2\gamma r_1}-r_1e^{-4\gamma r_1}$$

$$-\gamma r_1^2e^{-4\gamma r_1}]\,dr_1=e^2\gamma\left(\frac{5}{8}-2s\right)$$

Consequently, since $e^2\gamma=\frac{me^4}{\hbar^2}(2-s)$, we have

$$E=2E_0+\varepsilon=-\frac{me^4}{\hbar^2}(2-s)^2+\frac{me^4}{\hbar^2}(2-s)\left(\frac{5}{8}-2s\right)$$

$$=-\frac{me^4}{\hbar^2}(2-s)\left(\frac{11}{8}+s\right) \qquad (3)$$

From $\frac{\partial E}{\partial s}=0$ we find that

$$s_0=\frac{5}{16}$$

Hence, the energy of the ground state of helium is

$$E_{\min}=E(s_0)=-\left(\frac{27}{16}\right)^2\frac{me^4}{\hbar^2}$$

The ionization potential of the atom is then the difference between the obtained value $|E(s_0)|$ and the energy of a singly ionized atom of helium

$$|E_{\mathrm{He}^+}|=\frac{2^2me^4}{2\hbar^2}=\frac{2me^4}{\hbar^2}$$

i.e.

$$J=\frac{me^4}{\hbar^2}\left[\left(\frac{27}{16}\right)^2-2\right]=0.8477\,\frac{me^4}{\hbar^2}=23.0\text{ eV}$$

(The experimental value is $J=24.46$ eV.)

132. According to the conditions of the problem, we must find the approximate wave function that satisfies the equation

$$\Delta\psi+k^2\psi-\lambda U(r)\,\psi=0$$

where $k^2=2mE/\hbar^2>0$ (the scattering problem), and $\lambda U=2mV(r)/\hbar^2$ is considered small. For $\lambda=0$ the equa-

tion $\Delta\psi_0 + k^2\psi_0 = 0$ has a solution $\psi_0 = e^{i\mathbf{kr}}$ (the incident particle).

In the first approximation we look for $\psi = \psi_0 + \lambda\psi_1$. Identifying members with λ in the first power, we get the equation for ψ_1:

$$\Delta\psi_1 + k^2\psi_1 = Ue^{i\mathbf{kr}} \tag{1}$$

To find its solution we compare (1) with the equation for retarded potentials

$$\Delta\varphi - \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} = -4\pi\rho(\mathbf{r}, t)$$

whose solution is

$$\varphi(\mathbf{r}, t) = \int \frac{\rho\left(\mathbf{r}', t - \frac{|\mathbf{r}-\mathbf{r}'|}{c}\right)}{|\mathbf{r}-\mathbf{r}'|} d\tau$$

when $\rho = \rho_0 e^{-i\omega t}$ and $\varphi_0(\mathbf{r}, t) = \varphi_0(\mathbf{r})\, e^{-i\omega t}$. Then $\varphi_0(\mathbf{r})$ satisfies the equation

$$\Delta\varphi_0 + \frac{\omega^2}{c^2}\varphi_0 = -4\pi\rho_0 \tag{2}$$

and has a solution

$$\varphi_0(\mathbf{r}) = \int \frac{\rho_0(\mathbf{r}')\, e^{i\frac{\omega}{c}|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} d\tau'$$

Equation (2) for $\frac{\omega}{c} = k$ and $\rho_0 = -\frac{Ue^{i\mathbf{kr}}}{4\pi}$ coincides with Eq. (1). So for the scattered wave we can, obviously, write

$$\psi_{\text{scat}} = \lambda\psi_1 = -\frac{1}{4\pi}\int d\tau' \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\frac{2m}{\hbar^2} V(r')\, e^{i\mathbf{kr}'}$$

For $r \gg r'$ we have the approximate relationship $|\mathbf{r} - \mathbf{r}'| \approx \approx r - r' \cos\alpha$. We neglect r' compared with r in the denominator of the integrand, denote $kr' \cos\alpha = (\mathbf{k}_1 \cdot \mathbf{r}')$ ($\mathbf{k}_1$ points in the direction of the scattered wave), and introduce $|\mathbf{k} - \mathbf{k}_1| = K = 2k \sin\frac{\theta}{2}$, where θ is the scattering angle.

As a result we get

$$\psi_{\text{scat}} = -\frac{1}{4\pi}\frac{2m}{\hbar^2}\frac{e^{ikr}}{r}\int d\tau' e^{i(\mathbf{k}-\mathbf{k}_1)\mathbf{r}'}V(r')$$

$$= -\frac{2m}{4\pi\hbar^2}\frac{e^{ikr}}{r}\int_0^\infty V(r')\,r'^2\,dr' \times 2\pi\int_0^\pi e^{iKr'\cos\theta'}\sin\theta'\,d\theta'$$

We assume that $V(r')$ is a central potential. Since

$$\int_0^\pi e^{iKr'\cos\theta'}\sin\theta'\,d\theta' = 2\frac{\sin Kr'}{Kr'}$$

if we denote $f(\theta) = -\frac{2m}{\hbar^2}\int_0^\infty dr' r'^2 V(r')\frac{\sin Kr'}{Kr'}$, we can write for the scattered wave

$$\psi_{\text{scat}} = \frac{e^{ikr}}{r}f(\theta)$$

Composing the current densities by using the functions $\psi_0 = e^{i\mathbf{kr}}$ and ψ_{scat}, we get

$$d\sigma = \frac{|\mathbf{j}_{\text{scat}}\,dS|}{|\mathbf{j}_{\text{in}}|} = |f(\theta)|^2\frac{dS}{r^2} = |f(\theta)|^2\,d\Omega.$$

133. We compute $d\sigma$ for $V(r') = \frac{Ze^2}{r'}$ by using the results obtained in Problem 132. To make the integral converge, we temporarily introduce a factor $e^{-\alpha r'}$, i.e. we set $V(r') = \frac{Ze^2}{r'}e^{-\alpha r'}$. Then

$$f(\theta) = -\frac{2m}{\hbar^2}\frac{Ze^2}{K}\int_0^\infty e^{-\alpha r'}\sin Kr'\,dr' = -\frac{2m}{\hbar^2}\frac{Ze^2}{\alpha^2+K^2}$$

and for $\alpha = 0$, by substituting K, we get

$$f(\theta) = -\frac{2mZe^2}{\hbar^2K^2}$$

$$d\sigma = \frac{4m^2Z^2e^4}{\hbar^4K^4}\,d\Omega = \frac{m^2Z^2e^4\,d\Omega}{\hbar^4k^4\sin^2\frac{\theta}{2}}.$$

134. We write the energy of interaction between the particle that is colliding with the atom and the nucleus of that

atom (the charge of the nucleus being Ze) together with the electrons distributed around the nucleus with a density $\rho(r)$:

$$V(r') = \frac{Zee_1}{r'} - ee_1 \int \frac{\rho(r'')\,d\tau''}{|\mathbf{r}' - \mathbf{r}''|}$$

Hence, according to Problem 132, we write

$$f(\theta) = -\frac{mZee_1}{2\pi\hbar^2} I_1 + \frac{mee_1}{2\pi\hbar^2} I_2$$

where

$$I_1 = \int \frac{e^{i\mathbf{K}\mathbf{r}'}}{r'}\,d\tau' \quad \text{and} \quad I_2 = \int e^{i\mathbf{K}\mathbf{r}'}\,d\tau' \int \frac{\rho(r'')\,d\tau''}{|\mathbf{r}' - \mathbf{r}''|} \quad (\mathbf{K} = \mathbf{k} - \mathbf{k}_1)$$

To compute $I(\mathbf{r}'') = \int \frac{e^{i\mathbf{K}\mathbf{r}'}\,d\tau'}{|\mathbf{r}' - \mathbf{r}''|}$ we note that this integral satisfies the equation $\Delta I(\mathbf{r}'') = -4\pi e^{i\mathbf{K}\mathbf{r}''}$ and, therefore, if we set $I(\mathbf{r}'') = Ae^{i\mathbf{K}\mathbf{r}''}$, we get $-K^2Ae^{i\mathbf{K}\mathbf{r}''} = -4\pi e^{i\mathbf{K}\mathbf{r}''}$, i.e. $A = \frac{4\pi}{K^2}$. Thus,

$$I_1 = \frac{4\pi}{K^2}, \quad I_2 = 4\pi \int \rho(\mathbf{r}'')\, e^{i\mathbf{K}\mathbf{r}''} \frac{d\tau''}{K^2}$$

and

$$f(\theta) = -\frac{2mee_1}{\hbar^2 K^2} \{Z - F(\mathbf{K})\}$$

where

$$F(\mathbf{K}) = 4\pi \int_0^\infty \rho(r'') \frac{\sin Kr''}{Kr''} r''^2\,dr'' \quad \text{is the form factor} \qquad (1)$$

We set $\rho = \rho_0 e^{-r/a}$, and from the normalization condition

$$\int \rho\,d\tau = 4\pi\rho_0 \int_0^\infty e^{-r/a} r^2\,dr = 4\pi\rho_0 \times 2a^3 = Z$$

we can compute ρ_0:

$$\rho_0 = \frac{Z}{8\pi a^3}$$

Substituting this into (1), we get

$$F(\mathbf{K}) = 4\pi\rho_0 \int_0^\infty e^{-r/a} \frac{e^{iKr} - e^{-iKr}}{2iKr} r^2\, dr$$

$$= \frac{Z}{4ia^3K}\left[\frac{1}{\left(\frac{1}{a} - iK\right)^2} - \frac{1}{\left(\frac{1}{a} + iK\right)^2}\right] = \frac{Z}{(1+a^2K^2)^2}$$

and

$$d\sigma = \frac{4m^2e^2e_1^2}{\hbar^4K^4} Z^2 \left[1 - \frac{1}{(1+a^2K^2)^2}\right]^2 d\Omega$$

If we introduce $K = 2k \sin\frac{\theta}{2}$ and $\hbar k = mv$, we find

$$d\sigma = \frac{Z^2e^2e_1^2}{\left(\frac{mv^2}{2}\right)^2} \csc^4\frac{\theta}{2} \left[1 - \frac{1}{(1+a^2K^2)^2}\right] d\Omega$$

For fast particles that scatter at large angles, $aK \gg 1$ and

$$d\sigma \approx \frac{Z^2e^2e_1^2}{\left(\frac{mv^2}{2}\right)^2} \csc^4\frac{\theta}{2}\, d\Omega$$

For small aK we have $1 - \frac{1}{(1+a^2K^2)^2} \approx 2a^2K^2$ and

$$d\sigma \approx \frac{16m^2e^2e_1^2}{\hbar^4} Z^2a^4\, d\Omega$$

i.e. the differential cross section is finite.

135. According to the solution of Problem 132, $d\sigma = |f(\theta)|^2 d\Omega$, where $f(\theta)$ is the scattering amplitude in the form

$$f(\theta) = -\frac{2m}{\hbar^2} \int_0^\infty \frac{\sin Kr}{Kr} V(r)\, r^2\, dr \tag{1}$$

and $K = |\mathbf{k} - \mathbf{k}_1| = 2k \sin\frac{\theta}{2}$ (θ is the scattering angle).

Substituting the Yukawa potential into (1), we get

$$f(\theta) = -\frac{2mA}{\hbar^2K} \frac{1}{2i}\left[\int_0^\infty e^{-(\varkappa - iK)r}\, dr - \int_0^\infty e^{-(\varkappa + iK)r}\, dr\right]$$

$$= -\frac{mA}{\hbar^2Ki}\left[\frac{1}{\varkappa - iK} - \frac{1}{\varkappa + iK}\right] = -\frac{2mA}{\hbar^2(\varkappa^2 + K^2)}$$

and, hence,

$$d\sigma = \frac{4m^2A^2}{\hbar^4(\varkappa^2+K^2)^2}\, 2\pi \sin\theta\, d\theta$$

The total cross section is

$$\sigma = \int d\sigma = 2\pi \frac{4m^2A^2}{\hbar^4} \int_0^{\pi} \frac{\sin\theta\, d\theta}{[\varkappa^2 + 2k^2(1-\cos\theta)]^2}$$

$$= \frac{8\pi m^2A^2}{\hbar^4} \frac{1}{2k^2}\left(\frac{1}{\varkappa^2} - \frac{1}{\varkappa^2+4k^2}\right) = \frac{4}{\varkappa^2+4k^2}\left(\frac{2mA}{\varkappa\hbar^2}\right)^2.$$

SECTION IV

1. $p = \frac{1}{N}$.

2. $dw = \frac{2dt}{T} = \frac{1}{\pi}\frac{d\varphi}{\sqrt{\varphi_0^2-\varphi^2}}$.

3. $C = \frac{\alpha}{\pi}$.

4. $dw(x) = \sqrt{\frac{\alpha}{\pi}}\, e^{-\alpha x^2}\, dx$.

5. We use $P_n(t)$ to denote the probability of emission of n electrons in time t, and $P_0(t)$ to denote the probability of no emission of electrons in the same time. Assuming (1) and using the rule for calculating the probability of two consecutive events, we get

$$\begin{aligned} P_n(t+dt) &= P_{n-1}(t)\, P_1 + P_n(t)\,(1-P_1) \\ P_0(t+dt) &= P_0(t)\,(1-P_1) \end{aligned} \tag{1}$$

where P_1 is the probability of the emission of one electron in the time dt. According to condition (2), this probability is

$$P_1 = \lambda dt \tag{2}$$

Now we expand the left-hand sides of Eqs. (1) in a power series in dt and then tend dt to zero. We get

$$\begin{aligned} \frac{dP_n(t)}{dt} &= \lambda\,[P_{n-1}(t) - P_n(t)] \\ \frac{dP_0}{dt} &= -\lambda P_0(t) \end{aligned} \tag{3}$$

To this system of differential equations we must add the initial conditions for $P_n(t)$, namely

$$P_n(0)=\begin{cases} 1 & \text{for } n=0 \\ 0 & \text{for } n\neq 0 \end{cases} \tag{4}$$

It is easy to find the solution of (3) with initial conditions (4):

$$P_n(t)=\frac{(\lambda t)^n}{n!}e^{-\lambda t} \quad \text{(the Poisson distribution).}$$

6. Using the distribution obtained in Problem 5, we have

$$\overline{\Delta n^2}=\sum_{n=0}^{\infty} n^2P_n(t)-\left[\sum_{n=0}^{\infty} nP_n(t)\right]^2$$

$$=\lambda t\left[\lambda t\sum_{n=2}^{\infty}\frac{(\lambda t)^{n-2}}{(n-2)!}e^{-\lambda t}+\sum_{n=1}^{\infty}\frac{(\lambda t)^{n-1}}{(n-1)!}e^{-\lambda t}\right]$$

$$-(\lambda t)^2\left[e^{-\lambda t}\sum_{n=1}^{\infty}\frac{(\lambda t)^{n-1}}{(n-1)!}\right]^2=\lambda t$$

since $\sum_{n=0}^{\infty}\frac{x^n}{n!}=e^x$. But $\lambda t=\sum_{n=0}^{\infty} nP_n(t)=\bar{n}=n_0t$, and so $\overline{\Delta n^2}=n_0t$.

7. The probability that volume V_0 contains one molecule is expressed as $P=\frac{V_0}{V}$. Hence, the probability that any n molecules from the total N will land in V_0 is expressed as

$$P_n(V_0)=C_N^nP^n(1-P)^{N-n}$$

where $C_N^n=\frac{N!}{n!(N-n)!}$ is the number of ways in which we can choose any n molecules from the total N. The obtained formula

$$P_N(V_0)=\frac{N!}{(N-n)!\,n!}\left(\frac{V_0}{V}\right)^n\left(1-\frac{V_0}{V}\right)^{N-n} \tag{1}$$

is known as the binomial distribution.

We now consider the two extreme cases.

(a) Since $\bar{n} = PN$, we have

$$P_n = \lim_{N\to\infty} \frac{(\bar{n})^n}{n!}\left(1-\frac{\bar{n}}{N}\right)^N = \frac{(\bar{n})^n}{n!}e^{-\bar{n}} \tag{2}$$

This coincides with the final formula of the solution to Problem 5 if we remember that $\lambda t = \bar{n}$.

(b) Using Stirling's approximation $\ln n! \approx n\ln n - n$, we find from (2) that

$$\ln P_n = n\ln\bar{n} - \bar{n} - \ln n! = -(\bar{n}-\Delta n)\ln\left(1+\frac{\Delta n}{\bar{n}}\right) + \Delta n$$

Now bearing in mind that $\Delta n \ll \bar{n}$, we get the relationship for P_n,

$$\ln P_n \approx -\frac{(\Delta n)^2}{2\bar{n}}$$

correct to terms of the order of $O\left[\left(\frac{\Delta n}{\bar{n}}\right)^2\right]$. Hence,

$$P_n = Ce^{-\frac{(n-\bar{n})^2}{2\bar{n}}}$$

Normalizing this probability to unity, $\int_{-\infty}^{\infty} P_n\,dn = 1$, we find that

$$C = \frac{1}{\sqrt{2\pi\bar{n}}}$$

The expression obtained for P_n is called the Gaussian distribution.

8. The given inequality follows from the simple properties of the integral and the function $\rho(x)$:

$$w(x>a) = \int_a^x \rho(x)\,dx \leqslant \int_a^\infty \rho(x)\frac{x^2}{a^2}dx \leqslant \int_{-\infty}^\infty \rho(x)\frac{x^2}{a^2}dx \leqslant \frac{\overline{x^2}}{a^2}$$

Here $\rho(x)$ is the probability that the value of the random variable x lies in the interval $[x, x+dx]$.

9. We consider the following expression:

$$\frac{1}{2}(e^{i\varphi}+e^{-i\varphi}), \quad \text{where} \quad -\pi\leqslant\varphi\leqslant\pi$$

Here the coefficient of $e^{i\varphi}$ can be interpreted as the probability of moving one step to the right, and of $e^{-i\varphi}$ one step to the left. Clearly, the probability that after t steps the particle will reach point l will be equal to the coefficient of $e^{i\varphi l}$ in the binomial expansion

$$\left[\frac{1}{2}\left(e^{i\varphi}+e^{-i\varphi}\right)\right]^t=\frac{1}{2^t}e^{i\varphi t}+\dots$$
$$+P_t(l)\,e^{i\varphi l}+\dots+\frac{1}{2^t}e^{-i\varphi t} \quad (1)$$

Now we multiply (1) by $\frac{1}{2\pi}e^{-i\varphi l}$ and integrate the equation obtained with respect to φ from $-\pi$ to $+\pi$:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\frac{1}{2}\left(e^{i\varphi}+e^{-i\varphi}\right)\right]^t e^{-i\varphi l}\,d\varphi=P_t(l)$$

since

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{i\varphi(n-k)}\,d\varphi=\delta_{nk}=\begin{cases}1 & \text{for } n=k\\ 0 & \text{for } n\neq k\end{cases}$$

Hence, the sought probability will be

$$P_t(l)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\cos^t\varphi\times e^{-i\varphi l}\,d\varphi.$$

10. By analogy with Problem 9 we get the following results:

(1) for the square grating the probability that after t steps the particle will reach point $\mathbf{l}=(l_1,\ l_2)$ is

$$P_t(\mathbf{l})=\frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\left[\frac{1}{4}\left(e^{i\varphi_1}+e^{-i\varphi_1}+e^{i\varphi_2}+e^{-i\varphi_2}\right)\right]^t$$
$$\times e^{-i(\varphi_1 l_1+\varphi_2 l_2)}\,d\varphi_1\,d\varphi_2$$

(2) for the cubic grating this probability is

$$P_t(\mathbf{l})\equiv P_t(l_1,\ l_2,\ l_3)$$
$$=\frac{1}{(2\pi)^3}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\left[\frac{1}{3}\left(\cos\varphi_1+\cos\varphi_2+\cos\varphi_3\right)\right]^t$$
$$\times e^{-i(\varphi_1 l_1+\varphi_2 l_2+\varphi_3 l_3)}\,d\varphi_1\,d\varphi_2\,d\varphi_3.$$

11. Let us consider the totality of trajectories ending at point **l** and introduce the generating function

$$u_s(z, \mathbf{l}) = \sum_{t=0}^{\infty} P_t(l) z^t$$

$$= \frac{1}{(2\pi)^s} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \frac{e^{i(\varphi_1 l_1 + \varphi_2 l_2 + \ldots + \varphi_s l_s)}}{1 + \frac{z}{s} \cos\varphi_1 + \ldots + \cos\varphi_s)} d\varphi_1 \ldots d\varphi_s$$

where s is the dimension of the space.

Now we show that this function expresses the probability that the particle will not return to the origin of coordinates (the initial point). Let A be an event that can be repeated, f_j the probability that event A will take place for the first time in the jth trial, and u_j the probability that A will take place in the jth trial regardless of whether it occurred earlier.

We set $u_0 \equiv 1$ and construct the polynomials

$$u(z) = \sum_{j=0}^{\infty} u_j z^j, \quad F(z) = \sum_{j=1}^{\infty} f_j z^j$$

It is easy to see that

$$u_j = u_0 f_j + u_1 f_{j-1} + \ldots + u_{j-1} f_1$$

Multiplying both sides of the relationship and summing up with respect to j from 1 to ∞, we get

$$u(z) - 1 = F(z)\, u(z)$$

or

$$F(z) = 1 - [u(z)]^{-1}$$

But $F(1) = f_1 + f_2 + \ldots$ is the probability that event A will take place at some time.

There are thus two possibilities:

(a) if $u(1) = \infty$, then $F(1) = 1$ and A will certainly occur;

(b) if $u(1) < \infty$, then $F(1) < 1$ and there is a positive probability that event A will not take place.

In our case u (1) (the probability that the particle will at some time return to the initial point) coincides with

$$u_s(1) = \frac{1}{(2\pi)^s} \int_{-\pi}^{\pi} \dots \int_{-\pi}^{\pi} \frac{d\varphi_1 \dots d\varphi_s}{1 - \frac{1}{s}(\cos\varphi_1 + \dots + \cos\varphi_s)}$$

These integrals for the one-dimensional and two-dimensional cases diverge for small values of angle φ_i. For the one-dimensional case this is evident. For the two-dimensional case this can be proved if we pass to polar coordinates. Thus, in the one-dimensional and two-dimensional cases the particle in a random walk will always return to the initial point.

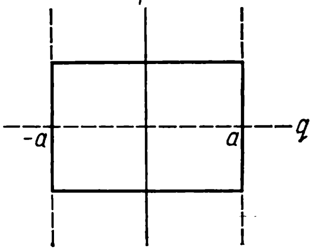

Fig. 59

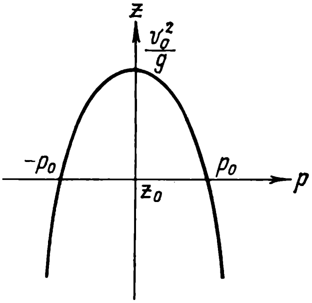

Fig. 60

In the three-dimensional case the integral can be evaluated numerically:

$$u_s(1) = \frac{1}{(2\pi)^3} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{d\varphi_1\, d\varphi_2\, d\varphi_3}{1 - \frac{1}{3}(\cos\varphi_1 + \cos\varphi_2 + \cos\varphi_3)} \approx 1.52$$

i.e. there is a certain probability (0.34) that the particle will not return to the initial point.

12. Since p is conserved, the phase trajectory has the form shown in Fig. 59.

13. The law of energy conservation gives us the equation for the phase trajectory:

$$\frac{p_0^2}{2m} + mgz_0 = \frac{p^2}{2m} + mgz$$

The trajectory is the parabola shown in Fig. 60.

14. $p = \pm\sqrt{2mee_1\left(\frac{1}{r} - \frac{1}{r_0}\right)}$.

15. We assume that at the initial time the velocity and the coordinate of the oscillator are equal to v_0 and x_0, res-

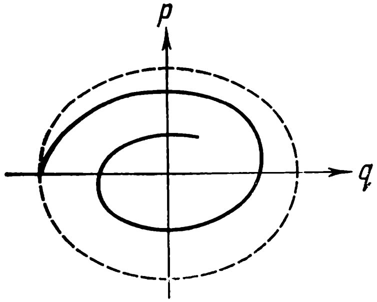

Fig. 61

pectively, and we have

$$x = e^{-\frac{\gamma t}{2}}\left[x_0 \cos \omega t + \frac{v_0}{\omega} \sin \omega t\right]$$

Since $\gamma \ll \omega_0$, we get

$$\frac{v}{\omega} = e^{\frac{-\gamma t}{2}}\left[\frac{v_0}{\omega} \cos \omega t - x_0 \sin \omega t\right]$$

Hence,

$$\left(x^2 + \frac{p^2}{k^2}\right) = \left(x_0^2 + \frac{p_0^2}{k^2}\right) e^{-\gamma t}$$

The equation for the trajectory represents an elliptical helix (Fig. 61). The change of phase volume with passage of time follows the law

$$\Gamma(t) = \iint\limits_G dp\, dx = \iint\limits_{G_0} \frac{\partial(p, x)}{\partial(p_0, x_0)} dp_0\, dx_0$$

$$= e^{-\gamma t} \iint\limits_{G_0} \begin{vmatrix} \cos \omega t & \frac{1}{mv} \sin \omega t \\ -mv \sin \omega t & \cos \omega t \end{vmatrix} dp_0\, dx_0 = e^{-\gamma t}\Gamma(0).$$

16. It follows from the law of conservation of energy that

$$\frac{p_\varphi^2}{2I} + mgL\,(1 - \cos\varphi) = H_0$$

where I is the moment of inertia, and p_φ is the generalized momentum. Hence,

$$p_\varphi = \pm\sqrt{2I\,(H_0 - mgL) + 2Img\cos\varphi}$$

Case (1). The phase trajectory is shown in Fig. 62. It

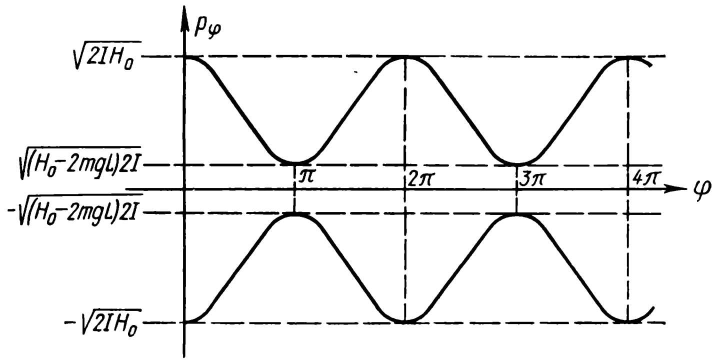

Fig. 62

corresponds to a rotational motion in two different directions.

Case (2). The phase trajectory is shown in Fig. 63. We can see that at point $\varphi = \pi$ the two branches of the phase

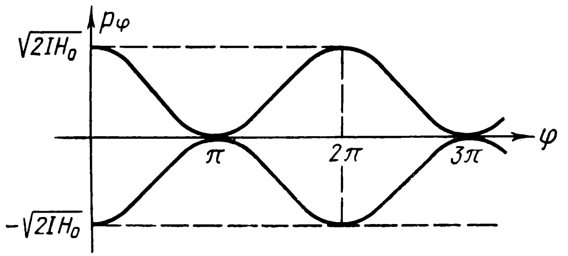

Fig. 63

trajectory come into contact with each other. An ambiguity appears at this point. But if we calculate the time needed

to reach point $\varphi = \pi$, we find that it is equal to infinity. Indeed, since $p_\varphi = I\,\frac{d\varphi}{dt}$, we have

$$t = \int_0^{\pi} \frac{I\,d\varphi}{(4mgLI\cos^2\varphi/2)^{1/2}} = \infty$$

Case (3). The possible values of angle φ will lie in the interval $[-\varphi_0, +\varphi_0]$, where φ_0 is determined from the condition $p_\varphi = 0$. The phase trajectories will have the form

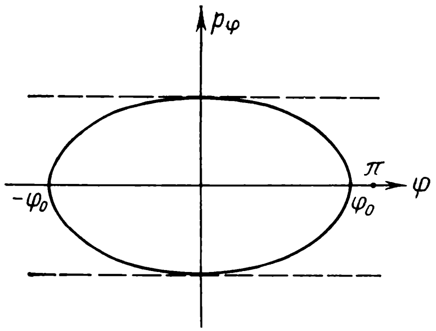

Fig. 64

shown in Fig. 64 and will describe the oscillations of the pendulum.

17. (1) The laws of energy and momentum conservation give us the connection between the momenta of the particles before and after collision:

$$p_1' = \frac{m_1 - m_2}{m_1 + m_2}\,p_1 + \frac{2m_1}{m_1 + m_2}\,p_2$$

$$p_2' = -\frac{m_1 - m_2}{m_1 + m_2}\,p_2 + \frac{2m_2}{m_1 + m_2}\,p_1$$

Hence, the Jacobian

$$\frac{\partial(p_1', p_2', q_1', q_2')}{\partial(p_1, p_2, q_1, q_2)} = 1$$

i.e. the phase volume is conserved.

(2) The triangle's vertices $A\,(p_0, z_0)$, $B\,(p_0, z_0 + a)$, $C\,(p_0 + b, z_0)$ will after a certain time t take the positions

$$A'\,(p_1,\ z_1),\ \text{where}\ p_1 = p_0 - mgt,\ z_1 = z_0 + \frac{p_0}{m}t - \frac{gt^2}{2}$$
$$B'\,(p_2,\ z_2),\quad \text{where}\quad p_2 = p_0 - mgt,\ z_2 = z_1 + a$$
$$C'\,(p_3,\ z_3),\ \text{where}\quad p_3 = p_1 + b,\ z_3 = z_1 + \frac{b}{m}t$$

(see Fig. 65).

Obviously, the area of the new triangle will be

$$S = \frac{ab}{2} = S_0$$

Thus, the area occupied by the phase points has changed in form but not in magnitude.

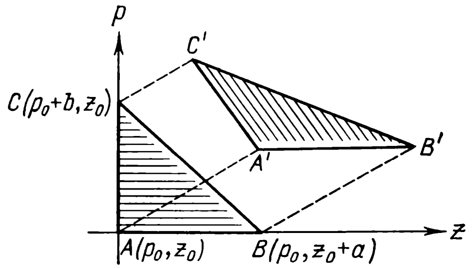

Fig. 65

18. The probability that the energy H of a closed system lies in the interval $[E,\ E + dE]$ follows from the microcanonical distribution given by formula (IV-4)

$$\rho\,(H) = \frac{\delta\,(H - E)}{\Omega\,(E)}$$

Here $\delta\,(H - E)$ is the Dirac delta function [see formula (1) in Appendix 4], and $\Omega\,(E)$ is the normalizing factor of the distribution:

$$\Omega\,(E) = \left.\frac{\partial\Gamma}{\partial H}\right|_{H=E}$$

where Γ is the phase volume limited by the hypersurface of constant energy $H\,(p_i,\ q_i) = F$. From this we find that

$$(1)\quad \Gamma\,(H) = \int \overset{3N}{\cdots} \int d^N p = \frac{\pi^{\frac{3N}{2}}}{\Gamma\left(\frac{3N}{2} + 1\right)}(2mH)^{\frac{3N}{2}}$$

where the integration was done over the hypersurface determined by the condition $\frac{p_1^2 + \ldots + p_N^2}{2m} \leqslant H$ [see formula (11) of Appendix 4]. This yields

$$\rho(H) = \delta(H-E)\,\frac{2}{3N}\,\frac{\Gamma\left(\frac{3N}{2}+1\right)}{2m^{\frac{3N}{2}}\,\pi^{\frac{3N}{2}}}\,E^{1-\frac{3N}{2}}$$

(2) $$\Gamma(H) = \int \overset{2N}{\ldots} \int d^N p\, d^N q = H^N \left(\frac{2m\pi}{\omega}\right)^N \frac{1}{N!}$$

where the integration was done over the hypersurface defined by the condition $\frac{p_1^2+\omega^2 q_1^2}{2m} + \ldots + \frac{p_N^2+\omega^2 q_N^2}{2m} \leqslant H$ [see formula (11) of Appendix 4]. This yields

$$\rho(H) = \delta(H-E)\,\frac{(N-1)!}{E^{N-1}}\left(\frac{\omega}{2\pi m}\right)^N.$$

19. If there is a thermal contact between the system with energy $H(p_i, q_i)$ $(i = 1, \ldots, n)$ and the heat bath with energy $H_0(p_i', q_i')$ $(i = 1, \ldots, N \gg n)$, which together form a closed system with energy $H + H_0 = E$, we can easily find the probability density for the ensemble from the microcanonical distribution. According to the law of addition of probabilities, we get

$$\rho(H) = \int \rho(H, H_0)\, dp_i'\, dq_i'$$

But $\rho(H, H_0)$ is given by the microcanonical distribution

$$\rho(H, H_0) = \frac{\delta(H+H_0-E)}{\Omega(E)}$$

So

$$\rho(H) = \frac{\Omega_0(E-H)}{\Omega(E)}$$

where $\Omega_0(E-H) = \left.\frac{\partial \Gamma_0}{\partial H_0}\right|_{H_0=E-H}$ and Γ_0 is the volume limited by the hypersurface of constant energy for the heat bath, $H_0(p_i', q_i') =$ constant. Using the results of Problem 18, we get for the two models the sought canonical distribution.

$$(1)\quad \Omega_0(E-H)=\frac{3N}{2}\,\frac{(2\pi m)^{\frac{3N}{2}}}{\Gamma\left(\frac{3N}{2}+1\right)}\,(E-H)^{\frac{3N}{2}-1}$$

We tend N to infinity but set

$$\lim_{\substack{E\to\infty\\ N\to\infty}}\frac{E}{N}=\frac{3}{2}kT$$

Then

$$\rho(H)=\lim_{N\to\infty}\frac{\frac{3N}{2}(2m\pi)^{\frac{3N}{2}}E^{\frac{3N}{2}-1}}{\Omega(E)}\lim_{N\to\infty}\left(1-\frac{H}{\frac{3}{2}NkT}\right)^{\frac{3N}{2}-1}$$

The first limit is finite since

$$\Omega(E)=\left.\frac{\partial\Gamma}{\partial(H+H_0)}\right|_{H+H_0=E}\geqslant\Omega_0(E)$$

Therefore,

$$\rho(H)=\text{constant}\times\lim_{N\to\infty}\left[\left(1-\frac{2H}{3NkT}\right)^{\frac{3NkT}{2H}}\right]^{H/kT}$$

$$=\text{constant}\times e^{-H/(kT)};$$

(2) The solution in this case is similar to case (1) but requires that

$$\lim_{N\to\infty}\frac{E}{N}=kT$$

The formulas obtained coincide with (IV-7), which can be derived in general form regardless of the specific model of the heat bath.

20. For a perfect gas

$$H=\sum_{i=1}^{N}[\varepsilon(\mathbf{p}_i)+U(\mathbf{r}_i)]=\sum_{i=1}^{N}H_i$$

where $\varepsilon(\mathbf{p}_i)$ is the kinetic energy of an individual molecule, and $U(\mathbf{r}_i)$ is the potential energy of interaction of a mole-

cule with the walls of the vessel:

$$U(\mathbf{r}_i) = 0 \quad \text{for} \quad \mathbf{r}_i \subset V,$$
$$= \infty \quad \text{for} \quad \mathbf{r}_i \not\subset V$$

Then

$$Z = \int \overset{\infty}{\underset{-\infty}{\cdots}} \int e^{-\sum_{i=1}^{N} H_i/kT} d^N p \, d^N r$$

$$= V^N \left[\int \overset{\infty}{\underset{-\infty}{\cdots}} \int e^{-\frac{H_i}{kT}} d\Gamma_i \right]^N = V^N (z_i)^N$$

Denoting $z_i \equiv f(T)$, we find the Helmholtz free energy

$$F = -NkT \ln V - NkT \ln f(T)$$

and then the entropy, pressure and average energy, respectively:

$$S = Nk \ln [f(T) V] + NkT \frac{\partial \ln f(T)}{\partial T}$$

$$p = \frac{NkT}{V}, \quad E = NkT^2 \frac{\partial \ln f(T)}{\partial T}.$$

21. (1) Assuming that $H_i = \frac{p_i^2}{2m}$, we get [see formula (13) of Appendix 4]:

$$z_i = (2\pi mkT)^{3/2}, \quad p = \frac{NkT}{V},$$

$$S = Nk \ln V + \frac{3}{2} Nk [\ln(2\pi mkT) + 1], \; E = \frac{3}{2} NkT, \; C_V = \frac{3}{2} kN.$$

(2) The energy in this case is

$$H_i = \frac{p_i^2}{2M} + \frac{1}{2\mu r_0^2} \left[p_{i\theta}^2 + \frac{p_{i\varphi}^2}{\sin^2\theta} \right]$$

$$z_i = (2\pi M kT)^{3/2} 8\pi^2 r_0^2 \mu \, kT = A (kT)^{5/2}$$

where $M = m_1 + m_2$, $\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}$, $A \equiv 8\pi^2 r_0^2 \mu (2\pi M)^{3/2}$, and r_0 is the distance between the atoms in the molecule. Then

$$p = \frac{NkT}{V}, \quad S = N \ln (VA) + \frac{5}{2} Nk \ln VkT + \frac{5}{2} Nk,$$

$$E = \frac{5}{3} NkT, \quad C_V = \frac{5}{2} Nk.$$

(3) For the approximation of small vibrations we have

$$H_i \approx \frac{p_i^2}{2M} + \frac{1}{2\mu}\left[p_{ir}^2 + \frac{p_{i\varphi}^2}{r^2 \sin^2\theta} + \frac{p_{i\theta}^2}{r^2}\right] + U(r_0) + \frac{\gamma}{2}(r-r_0)^2$$

where the potential energy of the atoms is represented as

$$U(r) = U(r_0) + \frac{1}{2}\frac{\partial^2 U}{\partial r^2}\bigg|_{r=r_0}(r-r_0)^2 + \ldots$$

$$\frac{\partial^2 U}{\partial r^2}\bigg|_{r=r_0} \equiv \gamma$$

At low temperatures we can easily estimate the classical partition function by bearing in mind that the integrand in (IV-10) has a sharp maximum at point r_0. Calculating the energy from $U(r_0)$, we find that

$$z_i = (2\pi M)^{3/2}\, 4\pi\,(2\pi\mu)^{3/2}\,(kT)^3 r_0^2 \sqrt{\frac{2\pi kT}{\gamma}} = B\,(kT)^{7/2}$$

where $B \equiv (2\pi M)^{3/2}\, 4\pi\,(2\pi\mu)^{3/2}\, r_0^2 \sqrt{\frac{2\pi}{\gamma}}$. Then

$$p = \frac{NkT}{V}, \quad S = Nk \ln V \cdot B + \frac{7}{2} Nk \ln(VkT) + \frac{7}{2} Nk,$$

$$E = \frac{7}{2} NkT, \quad C_V = \frac{7}{2} kN.$$

22. In both cases $p = \frac{NkT}{2}$ since the gases are perfect, and $H = \sum_{i=1}^{N} H_i$.

(a) $z_i = 4\pi \int_0^\infty e^{-\frac{ap^l}{kT}} p^2\, dp = \frac{4\pi}{l} \frac{\Gamma\left(\frac{3}{l}\right)}{a^{3/2}} (kT)^{3/2}$.

Hence, $E = \frac{1}{l} 3NkT$. In the given case $p = \frac{l}{3V} E$;

(b) $z_i = 4\pi \int_0^\infty e^{-\frac{c\sqrt{m^2c^2+p^2}}{kT}} p^2\, dp$. Substituting $mc \sinh t$ for p, we reduce z_i to

$$z_i = 4\pi m^3 c^3 \int_0^\infty e^{-z_0 \cosh t} \sinh^2 t \cosh t\, dt$$

where $z_0 \equiv \frac{mc^2}{kT}$. Hence,

$$z_i = \frac{4\pi}{z_0} m^3 c^3 \left[K_0(z_0) + \frac{2}{z_0} K_1(z_0) \right]$$

where $K_0(z_0)$ and $K_1(z_0)$ are modified Hankel functions of order 0 and 1. The system's average energy is then

$$E = NkT \left[1 + 2 \frac{K_0(z_0) + \left(\frac{z_0}{2} + \frac{2}{z_0} \right) K_1(z_0)}{K_0(z_0) + \frac{2}{z_0} K_1(z_0)} \right]$$

Using the asymptotic expansions for the modified Hankel functions, we see that for $kT \ll mc^2$,

$$E = Nmc^2 + \frac{3}{2} NkT.$$

23. Let the minimum value of the system's energy be equal to zero. Then

$$Z(\beta) = \int_0^\infty e^{-\beta E} \, \Omega(E) \, dE \tag{1}$$

If we assume that $E > 0$ and that for $E < 0$, $\Omega(E) = 0$ we find

$$\Omega(E) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} e^{\beta E} Z(\beta) \, d\beta$$

because it follows from formula (1) that Z and $\Omega(E)$ are connected via the Laplace transformation. Now, using the Cauchy theorem for the derivative, we find

$$\Omega(E) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{A}{\beta^N} e^{\beta E} \, d\beta = AE^{N-1} \frac{1}{(N-1)!}.$$

24. $\rho = C \exp \left\{ -\frac{H(\mathbf{v}_i', \mathbf{r}_i')}{kT} + \frac{1}{2kT} \sum_i m [\Omega \times \mathbf{r}_i']^2 \right\},$

where $\mathbf{v}_i'$ and $\mathbf{r}_i'$ are, respectively, the velocities and radius vectors of the particles in the revolving system of coordinates.

25. Using the result of Problem 24, we get

$$F = F_0 - NkT \ln\left[\frac{2kT}{m\Omega^2 R^2}\left(e^{\frac{m\Omega^2 R^2}{2kT}} - 1\right)\right]$$

Recalling that $dV = 2\pi hR\,dR$, we find

$$p = -\frac{NkT}{V}\,\frac{U(R)}{kT}\,\frac{e^{-U/kT}}{e^{-U/kT}-1}, \quad U \equiv -\frac{m\Omega^2 R^2}{2}.$$

26. We consider the following total time derivative of the sum over all the particles in the system:

$$\frac{d}{dt}\sum_{i=1}^{N} \mathbf{r}_i \mathbf{p}_i = \sum_{i=1}^{N} \mathbf{p}_i \frac{\partial T}{\partial \mathbf{p}_i} + \sum_{i=1}^{N} \mathbf{r}_i \dot{\mathbf{p}}_i \tag{1}$$

Next we find its time average

$$\widetilde{F}^t = \lim_{t\to\infty} \frac{1}{t}\int_0^t F(t')\,dt' \tag{2}$$

Now, if the particles move in a finite region and with velocities that do not turn into infinity in this region, the time average of the left-hand side of (1) will give zero ($t \to \infty$). For the terms in the right-hand side we use Euler's theorem on homogeneous functions and get

$$\sum_{i=1}^{N} \widetilde{\mathbf{p}_i \frac{\partial T}{\partial \mathbf{p}_i}} = 2\widetilde{T} \equiv 2T_0$$

where T_0 is the system's average kinetic energy.

Now let us find the time average of the second term

$$\sum_{i=1}^{N} \mathbf{r}_i \dot{\mathbf{p}}_i = \sum_{i=1}^{N} \mathbf{r}_i \mathbf{F}_i$$

where $\mathbf{F}_i$ is the force acting on the ith particle. The sum $\sum_{i=1}^{N} \mathbf{r}_i \mathbf{F}_i$ is called the virial in mechanics.

Let the particles of the system move in volume V. Then a force $-\mathbf{n}p(t)$ will act on the particles from each unit area ($\mathbf{n}$ is the external normal to the surface that encloses volume

V and $p(t)$ is the pressure at the given time that the particles exert on the unit area). Since, besides this, forces of interaction $\mathbf{F}_{ij} = -\mathbf{F}_{ji}$ act between the particles, we get

$$\sum_{i=1}^{N} \overline{\mathbf{r}_i \mathbf{F}_i} = -\oiint \overline{\mathbf{nr}p}\, dS - \sum_{i=1}^{N} \overline{\mathbf{r}_i \frac{\partial U}{\partial \mathbf{r}_i}} = -p_0 \cdot 3V - nU_0$$

since

$$\oiint_S \mathbf{nr}\overline{p(t)}\, dS = \overline{p(t)} \iiint_V \operatorname{div} \mathbf{r}\, dV = 3p_0 V$$

where p_0 is the average pressure in the system, and $U_0 = \frac{1}{2} \sum_{i,j} \overline{U(\mathbf{r}_i, \mathbf{r}_j)}$ is the average potential energy of interaction between the particles.

So in the final analysis we find that the pressure in the system is

$$p = \frac{2}{3} \frac{T_0}{V} + \frac{nU_0}{3V} = \frac{2}{3} \frac{N\varepsilon_0}{V} + \frac{nU_0}{3V}$$

where ε_0 is the average kinetic energy per particle.

27. $\overline{H^n} = (kT)^n \dfrac{\Gamma\left(\frac{3N}{2} + n\right)}{\Gamma\left(\frac{3N}{2}\right)}$ [see formula (14) of Appendix 4]. We can easily find now that

$$\overline{\Delta H^2} = \overline{(H - \overline{H})^2} = \frac{3N}{2} (kT)^2, \quad \delta^2 = \frac{2}{3} \frac{1}{N}.$$

28. Let

$$H = \sum_{i=1}^{N} \frac{p_i^2}{2m} + U(\mathbf{r}_1, \ldots, \mathbf{r}_N)$$

Then, using the theorem on addition of probabilities, we can find the probability that the velocity of a particle will be

in the interval $[\mathbf{v}, \mathbf{v}+d\mathbf{v}]$. So for case (1) we find that

$$d\rho(\mathbf{v}) = e^{-\frac{mv^2}{2kT}} \int\limits_{-\infty}^{\infty} \cdots \int e^{-\sum\limits_{i=1}^{N} \frac{mv_i^2}{2kT}} d^{N-1}\mathbf{v}$$

$$\times \int\limits_{-\infty}^{\infty} \cdots \int e^{-\frac{U(\mathbf{r}, \ldots, \mathbf{r}_N)}{kT}} d^N r \, dv_x \, dv_y \, dv_z$$

$$= Ce^{-mv^2/(2kT)} dv_x \, dv_y \, dv_z$$

where $C = \left(\frac{m}{2\pi kT}\right)^{3/2}$ is found from the normalization condition

$$\iiint\limits_{-\infty}^{\infty} d\rho(\mathbf{v}) = 1$$

Recalling that $dv_x \, dv_y \, dv_z = v^2 \sin^2\theta \, dv \, d\theta \, d\varphi$ and $\varepsilon = mv^2/2$, we get for case (2)

$$d\rho(v) = 4\pi \left(\frac{m}{2\pi kT}\right)^{3/2} e^{-mv^2/(2kT)} v^2 \, dv$$

and for case (3)

$$d\rho(\varepsilon) = \frac{2}{\sqrt{\pi (kT)^3}} e^{-\varepsilon/kT} \sqrt{\varepsilon} \, d\varepsilon.$$

29. We follow the general rule of finding the mean and get

$$\overline{v^n} = \left(\frac{m}{2\pi kT}\right)^{3/2} 4\pi \int\limits_0^{\infty} v^{n+2} e^{-\frac{mv^2}{2kT}} dv = \frac{4}{\sqrt{\pi}} \left(\frac{2kT}{m}\right)^{n/2} \Gamma\left(\frac{n+3}{2}\right)$$

Now we use the value of the gamma function [see formulas (5) and (6) of Appendix 4] and find that

$$\bar{v} = \sqrt{\frac{8kT}{\pi m}}, \quad \overline{v^2} = \frac{3kT}{m}$$

And from the condition $\frac{\partial}{\partial v}(e^{-mv^2/(2kT)} v^2) = 0$ we find that

$$v_0 = \sqrt{\frac{2kT}{m}}.$$

30. It follows from the formulas of Problem 29 that

$$\bar{\varepsilon}=\frac{m\overline{v^2}}{2}=\frac{3}{2}kT, \quad \varepsilon_0=\frac{mv_0^2}{2}=\frac{kT}{2}$$

Obviously, $\bar{\varepsilon}\neq\varepsilon_0$, although the following relationship holds for the energy of the entire system:

$$H_0\approx\bar{H}=E$$

This relationship can hold only if there are a great number of particles in the entire system.

31. We write the Maxwell momentum distribution and get

$$d\rho(p)=Ce^{-\frac{c\sqrt{m^2c^2+p^2}}{kT}}\,dp_x\,dp_y\,dp_z$$

The normalization condition $\int d\rho(p)=1$ yields

$$C=\left\{4\pi(mc)^3\left[2\left(\frac{kT}{mc^2}\right)K_1\left(\frac{mc^2}{kT}\right)+\frac{kT}{mc^2}K_0\left(\frac{mc^2}{kT}\right)\right]\right\}^{-1}$$

where $K_0(z)$ and $K_1(z)$ are modified Hankel functions.

32. $d\rho(\mathbf{v},\mathbf{u})=\left(\frac{m}{2\pi kT}\right)^{3/2}e^{-\frac{m}{2kT}(\mathbf{v}-\mathbf{u})^2}\,dv_x\,dv_y\,dv_z.$

33. We determine the probability that the first particle has a velocity $\mathbf{v}_1$, and the second $\mathbf{v}_2$. The canonical distribution yields

$$d\rho(\mathbf{v}_1,\mathbf{v}_2)=ce^{-\frac{m_1v_1^2+m_2v_2^2}{2kT}}\,d\mathbf{v}_1\,d\mathbf{v}_2$$

To find c we use the normalization condition, and we get

$$d\rho(v_1,v_2)=16\pi^2\left(\frac{m_1m_2}{4\pi^2k^2T^2}\right)^{3/2}e^{-\frac{m_1v_1^2+m_2v_2^2}{2kT}}\,v_1^2v_2^2\,dv_1\,dv_2$$

Now we change variables, $\mathbf{v}'=\mathbf{v}_1-\mathbf{v}_2$ and $\mathbf{v}_0=\frac{m_1\mathbf{v}_1+m_2\mathbf{v}_2}{m_1+m_2}$, and using the fact that $m_1m_2=\mu M$, we get

$$d\rho(v',v_0)=4\pi\left(\frac{\mu}{2\pi kT}\right)^{3/2}e^{-\frac{\mu v'^2}{2kT}}v'^2\,dv'\times4\pi\left(\frac{M}{2\pi kT}\right)^{3/2}\times e^{-\frac{Mv_0^2}{2kT}}v_0^2\,dv_0$$

$$\left(M=m_1+m_2,\ \frac{1}{\mu}=\frac{1}{m_1}+\frac{1}{m_2}\right)$$

We integrate this expression with respect to the centre-of-mass velocity v_0 and thus find the relative velocity distribution of the particles:

$$d\rho(v') = 4\pi \left(\frac{M}{2\pi kT}\right)^{3/2} e^{-\frac{Mv'^2}{2kT}} v'^2 \, dv'$$

It follows from this that

$$\overline{v'} = \sqrt{2}\,\bar{v}$$

where $\bar{v}$ is the mean velocity of the particles of the gas.

34. To characterize the collision of particles we introduce the total scattering cross section, σ, which is the ratio of the probability of the given collision in unit time to the flux of the particles per unit area. If the mean number of particles in unit volume is n, the average number of collisions of one molecule with all the others per unit time will be

$$\nu = \int \sigma(v')\, nv' \, d\rho(v')$$

For our case $\sigma = 4\pi R_0^2$, then

$$\nu = 4\pi\sigma \sqrt{\frac{kT}{\pi m}}$$

Hence,

$$\lambda = \frac{\bar{v}}{\nu} = \frac{1}{4\pi n R_0^2 \sqrt{2}}.$$

35. $$\frac{n_2}{n_1} = \frac{\frac{\sqrt{\pi}}{2}\operatorname{erf}(1) - \frac{1}{e}}{\frac{\sqrt{\pi}}{2} + \frac{1}{e} - \frac{\sqrt{\pi}}{2}\operatorname{erf}(1)},$$ where $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \times$ $\times \int_0^x e^{-t^2}\, dt$ is the error integral [see formulas (15)-(18) in Appendix 4]. But $\operatorname{erf}(1) = 0.84$, and so $n_2/n_1 = 0.4$.

36. The total scattering cross section in this case is

$$\sigma = \pi D^2 \left[1 + \frac{\alpha}{D^n m} \frac{1}{v'^2}\right]$$

where $D = 2R_0$, and v' is the relative velocity of motion of the particles. We average this expression with respect to

v' and get the Sutherland relation

$$\bar{\sigma} = \pi D^2 \left(1 + \frac{a}{T}\right)$$

where $a \equiv \dfrac{\alpha}{D^n k}$.

37. $\nu = n_0 \sigma \bar{v} \sqrt{2}$. Here $n_0' = \dfrac{N}{S}$, $\bar{v}$ is the mean velocity in the two-dimensional case, N is the total number of particles on the surface, S is the surface area.

38. Since the atoms move with different velocities, an observer will see light of all wavelengths due to the Doppler effect. For example, if the atom is moving away from the observer with a velocity v_z and the observer is on the z-axis, the light will seem to him to have the following wavelength:

$$\lambda = \lambda_0 \left(1 + \frac{v_z}{c}\right)$$

And so the intensity of light seen by the observer in the interval from λ to $\lambda + d\lambda$ will be

$$J d\lambda = \alpha \; dn(\lambda)$$

where $dn(\lambda)$ is the number of atoms in the entire volume that radiate light with wavelengths lying in $[\lambda, \lambda + d\lambda]$, and α is a constant that is determined by the condition

$$\int J(\lambda)\, d\lambda = N J_0$$

If we assume the Maxwell velocity distribution to be valid in our case, we have

$$dn(\lambda) = dn(v_z) = N \left(\frac{m}{2\pi kT}\right)^{1/2} e^{-m v_z^2/(2kT)}\, dv_z$$

$$= N \left(\frac{mc^2}{2\lambda_0^2 \pi kT}\right)^{1/2} e^{-\frac{mc^2}{2kT\lambda_0^2}(\lambda - \lambda_0)^2}\, d\lambda$$

since

$$v_z = \frac{c}{\lambda_0}(\lambda - \lambda_0)$$

Then

$$J(\lambda)\, d\lambda = \alpha N \frac{1}{\sqrt{\pi \delta^2}}\, e^{-(\lambda - \lambda_0)^2/\delta^2}\, d\lambda$$

where $\delta = \sqrt{\frac{2kT/\lambda_0^2}{mc^2}}$ is the Doppler half-width of the spectral line. We now find α:

$$NJ_0 = \frac{\alpha N}{\sqrt{\pi\delta^2}} \int_0^{\infty} e^{-(\lambda-\lambda_0)^2/\delta^2}\, d\lambda$$

$$\approx \frac{\alpha N}{\sqrt{\pi\delta^2}} \int_{-\infty}^{\infty} e^{-(\lambda-\lambda_0)^2/\delta^2}\, d\lambda = \alpha N$$

$$\alpha = J_0$$

where the lower limit of integration has been changed to $-\infty$ since the integrand for $\lambda < 0$ is practically zero.

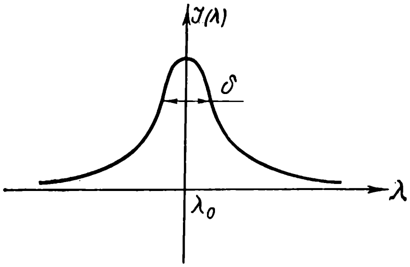

Fig. 66

Thus, the spectral density is represented by the Gaussian distribution (Fig. 66):

$$J(\lambda) = \frac{J_0 N}{\sqrt{\pi}\,\delta} e^{-(\lambda-\lambda_0)^2/\delta^2}.$$

39. If we assume that the energy of a free electron inside the metal is less than its energy outside the metal by the work function $e\varphi$ and that the electrons obey the Maxwell distribution, we find the current density along the x-axis that is perpendicular to the surface of the metal:

$$j_x = n_0 e \left(\frac{m}{2\pi kT}\right)^{3/2} \int_{v_{0x}}^{\infty} v_x e^{-\frac{mv_x^2}{2kT}}\, dv_x \iint_{-\infty}^{\infty} e^{-\frac{mv_y^2+mv_z^2}{2kT}}\, dv_y\, dv_z$$

The value of v_{0x} can be determined from the condition that the electron leaves the metal, i.e.

$$\frac{mv_{0x}^2}{2} = e\varphi$$

Then

$$j_x = \frac{n_0 e\bar{v}}{4} e^{-e\varphi/(kT)}$$

which is the classical Richardson equation.

41. Integrating the canonical distribution with respect to momentum, we find the sought probability:

$$d\rho(\mathbf{r}) = Ce^{-\frac{U(x, y, z)}{kT}} dx\, dy\, dz$$

The number of these particles in unit volume is

$$dn(\mathbf{r}) = n_0 d\rho(\mathbf{r})$$

where n_0 is the average number of particles in unit volume. This is the Boltzmann distribution.

42. By definition we have

$$z_0 = \frac{\int\limits_0^\infty ze^{-\frac{mgz}{kT}} dz}{\int\limits_0^\infty e^{-\frac{mgz}{kT}} dz} = \frac{kT}{mg}$$

since the Boltzmann distribution yields

$$dm(z) \propto e^{-\frac{mgz}{kT}}.$$

43. The centre of gravity for one type of particles is

$$z_k = \frac{\int\limits_0^h ze^{\frac{m_k gz}{kT}} dz}{\int\limits_0^h e^{-\frac{m_k gz}{kT}} dz} = z_{0k} - \frac{h}{e^{\frac{h}{z_{0k}}} - 1}$$

where $z_{0k}=\frac{kT}{m_k g}$. The centre of gravity of the entire system will be

$$z_0=\frac{\sum_{k=1}^{l} N m_k z_k}{\sum_{k=1}^{l} N m_k}=\frac{lkT}{gM}-\frac{1}{M}\sum_{k=1}^{l}\frac{h m_k}{e^{\frac{h}{z_{0k}}}-1}$$

$$\left(M=\sum_k m_k\right).$$

45. $M=0.41\,\frac{SP_2}{2\pi mkT_2}$.

46. We introduce spherical coordinates by directing the z-axis along vector $\mathbf{u}$ (Fig. 67). If the particle collides with the sphere at point A with a relative velocity $\mathbf{v}$, we get $|\mathbf{v}_1'|=|\mathbf{v}'|$ and $\gamma=\gamma'$ (an elastic collision). It follows from this that the sphere will experience "resistance" from the impact with one particle equal to

$$-2mv'\cos\gamma\cos\alpha \qquad (1)$$

since the sphere receives momentum equal to $2mv'\cos\gamma$ directed along OB. But

$$\cos\alpha=\cos\theta\cos\gamma+\sin\theta\sin\gamma\cos\beta \qquad (2)$$

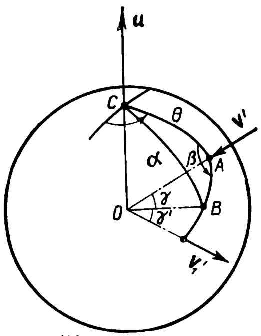

Fig. 67

(the law of cosines for the spherical triangle ABC).

The Maxwell distribution in the given coordinates is

$$dn(v')=n\left(\frac{m}{2\pi kT}\right)^{3/2}\times\exp\left\{-\frac{m}{2kT}(v'^2+u^2-2v'u\cos\theta)\right\}v'^2\sin\theta\,dv'\,d\theta\,d\varphi \qquad (3)$$

and the number of collisions in unit time, with point B (γ, β) lying in the solid angle differential $\sin\gamma\,d\gamma\,d\beta$, is equal to the number of particles in volume

$$R^2\sin\gamma\,d\gamma\,d\beta\,v'\cos\gamma \qquad (4)$$

(an area element multiplied by path per unit time).

Keeping in mind the symmetry of the problem and using relations (1)-(4), we get the total force of resistance

$$F=-2mn\left(\frac{m}{2\pi kT}\right)^{3/2}R^2\int_0^\infty v'^4\,dv'\int_0^\pi 2\pi\sin\theta\,d\theta$$

$$\times\exp\left[-\frac{m}{2kT}(v'^2+u^2-2v'u\cos\theta)\right]\int_0^{\pi/2}\cos^2\gamma\sin\gamma\,d\gamma$$

$$\times\int_0^{2\pi}d\beta\,(\cos\theta\cos\gamma+\sin\theta\sin\gamma\cos\beta)$$

$$=2\pi^2R^2m\left(\frac{m}{2\pi kT}\right)^{3/2}\frac{(kT)^2}{mu^2}\int_0^\infty v'^2\left[\left(\frac{mu}{kT}v'-1\right)e^{\frac{-m(u-v')^2}{2kT}}\right.$$

$$\left.+e^{-\frac{m(u+v')^2}{2kT}}\left(\frac{mu}{kT}v'+1\right)\right]dv'$$

Using the relationships (15)-(18), Appendix 4, for the error integral, we find the final form for F:

$$F=-nkT\pi R^2\left[\frac{e^{-b^2}}{b\pi^{1/2}}(1+2b^2)+\left(2b^2+2-\frac{1}{2b^2}\right)\Phi(b)\right]$$

where $b=\dfrac{u}{\sqrt{m/2kT}}$ and $\Phi(b)=\dfrac{2}{\sqrt{\pi}}\displaystyle\int_0^b e^{-x^2}\,dx$. The order of magnitude of b is determined by the ratio of the velocity of the sphere to the mean velocity of motion of the particles. We consider the case for $b\ll 1$. The expansion

$$\Phi(b)=\frac{2}{\sqrt{\pi}}\left(b-\frac{b^3}{3}+\ \dots\right)$$

then gives

$$F=-6\pi\alpha Rv$$

where

$$\alpha=\frac{8}{9}nR\sqrt{\frac{mkT}{2\pi}}$$

We compare this expression with Stokes' law for the force of resistance experienced by a sphere moving in viscous

liquid:

$$F = -6\pi\eta\, Rv$$

where η is the coefficient of viscosity, and we find that for small velocities of the sphere these formulas have a qualitative similarity.

47. $N = \int_0^{\theta_0} dN(\theta) = S\frac{N}{2V}\sqrt{\frac{2kT}{\pi m}}\frac{R^2}{R^2+h^2}$ (see Fig. 46).

48. Using the Maxwell distribution, we find the number of molecules that pass through the aperture S_0 in time dt:

$$-dN = dt\ n_0 S_0 \int_0^\infty v_x\, dv_x \int_{-\infty}^{\infty}\int e^{-\frac{mv^2}{2kT}} dv_y\, dv_z \left(\frac{m}{2\pi kT}\right)^{3/2}$$

$$= n_0 S_0 \sqrt{\frac{kT}{2\pi m}}\, dt = \frac{p}{kT} S_0 \sqrt{\frac{kT}{2\pi m}}\, dt$$

The velocity of outflow of gas will be

$$v = -\frac{dN}{dt} = \frac{pS_0}{4kT}\bar{v}.$$

49. Keeping in mind the relationship $\bar{A}_s = kT\left(\frac{\partial \ln Z}{\partial a_s}\right)_T$, we find that the mean dipole moment is

$$P = -kT\left(\frac{\partial \ln Z}{\partial E}\right)_T$$

where

$$Z = (z_i)^N, \quad z_i = \int e^{-\frac{H_i}{kT}} d\Gamma, \quad H_i = H_{0i} - (\mathbf{p}_0\cdot\mathbf{E})$$

Here H_{0i} is the energy of one particle without the field. Hence (see the solution to Problem 21),

$$z_i = \varphi(T)\int_0^\pi e^{\frac{p_0 E\cos\theta}{kT}} \sin\theta\, d\theta$$

where $\varphi(T)$ is a known function of temperature. Then

$$P = Np_0\left[\coth\frac{p_0E}{kT} - \frac{kT}{p_0E}\right] = Np_0 L\left(\frac{p_0E}{kT}\right)$$

where $L(x) = \coth x - \frac{1}{x}$ is the Langevin function.

In the case of high temperatures or weak electric fields

$$\frac{p_0E}{kT} \ll 1$$

Then expanding $L\left(\frac{p_0E}{kT}\right)$ in a power series

$$L\left(\frac{p_0E}{kT}\right)=\frac{1}{3}\frac{p_0E}{kT}+O\left[\left(\frac{p_0E}{kT}\right)^3\right]$$

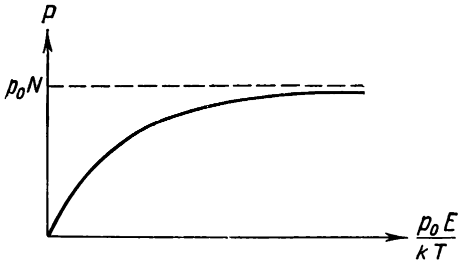

Fig. 68

we get a linear dependence of P on E (Fig. 68):

$$P=\frac{1}{3}Np_0\frac{p_0E}{kT}$$

The polarization is then

$$p=\frac{P}{V}=\frac{N}{3V}\frac{p_0^2E}{kT}=\beta E$$

where $\beta=\frac{1}{3}\frac{N}{V}\frac{p_0^2}{kT}$ is, by definition, the polarizability of the gas. The permittivity is then

$$\varepsilon=1+4\pi\beta=1+\frac{4}{3}\frac{\pi np_0^2}{kT}\quad\left(n=\frac{N}{V}\right).$$

50. $\varepsilon=1+4\pi\left(n\alpha+\frac{1}{3}\frac{np_0^2}{kT}\right)$.

51. Choose the z-axis along the vector of magnetic induction $\mathbf{B}$ of the external field, i.e. $\mathbf{B}=0\cdot\mathbf{i}+0\cdot\mathbf{j}+B_0\cdot\mathbf{k}$. The vector potential will then be

$$\mathbf{A}=0\cdot\mathbf{i}+B_0\cdot x\cdot\mathbf{j}+0\cdot\mathbf{k}$$

and the Lagrangian will be

$$\mathcal{L}(r, v)=\sum_{i=1}^{N}\frac{m_i v_i^2}{2}-U(\mathbf{r}, \ldots, \mathbf{r}_N)+\sum_{k\neq i}\frac{e_i e_k}{r_{ik}}(\mathbf{v}_i\cdot\mathbf{v}_k)+\sum_{k=1}^{N} e_k(\mathbf{v}_k\cdot\mathbf{A})$$

Hence,

$$Z=\int_{-\infty}^{\infty}\ldots\int \exp\left\{-\frac{1}{kT}\left[\sum_{k=1}^{3N}\frac{\partial L}{\partial \dot{r}_k}\dot{r}_k-L\right]\right\} d^N\mathbf{r}\, d^N\mathbf{p}$$

$$=\int_{-\infty}^{\infty}\ldots\int \exp\left\{-\frac{1}{kT}\left[\sum_{k=1}^{3N}\frac{\partial L}{\partial \dot{r}_k}\dot{r}_k-L\right]\right\} D_{ik}\, d^N\mathbf{r}\, d^N\mathbf{v}$$

where the determinant

$$D_{ik}=\begin{vmatrix}\frac{\partial p_{1x}}{\partial v_{1x}}, & \ldots, & \frac{\partial p_{1x}}{dv_{Nz}}\\ \cdots & \cdots & \cdots\end{vmatrix}=\begin{vmatrix}\frac{\partial^2 L}{\partial v_{1x}\,\partial v_{1x}}, & \ldots, & \frac{\partial^2 L}{\partial v_{1x}\,\partial v_{Nz}}\\ \cdots & \cdots & \cdots\end{vmatrix}$$

does not depend on B because in the terms with B, $\mathcal{L}$ is linear in $\mathbf{v}$. The exponent (the expression in braces) does not depend on B either, because terms with B will cancel out. Hence, the magnetization vector along the field is

$$M_z=-kT\frac{\partial \ln Z}{\partial B}=0$$

which means that a classical system cannot be magnetized stably. This appears paradoxical because a "classical" explanation is often given for dia- and paramagnetism. This "classical" explanation assumes the existence of stable electrical systems with stable electron orbits. But this is the very thing that is not explicable from the standpoint of classical physics.

52. $\varkappa=\sqrt{\dfrac{T_i T_e}{4\pi n_0 e^2(T_i-T_e)}}$.

53. The Hamiltonian in this case has the form

$$H=H(p_i, q_i)+\frac{P_M^2}{2M}+Mgz$$

where P_M and z are the momentum and coordinate of the piston. Since the relationship $Mg = pS$ must always hold, the canonical distribution takes the form

$$\rho = c\exp\left\{-\frac{H(p_i, q_i) - \frac{P_M^2}{2M} - pV}{kT}\right\}$$

We find the classical partition function for this case:

$$Z = \int_{-\infty}^{\infty} e^{-\frac{P_M^2}{2MkT}} dP_M \int_0^{\infty} e^{-\frac{pV}{kT}} dV\,(2\pi mkT)^{\frac{3N}{2}} V^N$$

$$= N!\,(2\pi m)^{\frac{3N+1}{2}} \sqrt{\frac{M}{m}}\,(kT)^{\frac{5N+3}{2}} p^{-(N+1)}$$

and the mean volume of the system is

$$\overline{V} = \int V\rho(V)\,d\Gamma = \left(\frac{\partial F}{\partial p}\right)_T = \frac{N+1}{p} kT$$

This relationship determines the equation of state when the volume varies.

54. *Hint.* Use the property of additivity for the Hamiltonian.

55. We consider an arbitrary mechanical quantity $U(p_i, q_i, a_s)$. Differentiating the mean value of $\overline{U}$ with respect to T, we get

$$\frac{\partial \overline{U}}{\partial T} = -\frac{1}{kT^2}\overline{(U-\overline{U})(H-\overline{H})}$$

We set $U = H$ and find that

$$\overline{(H-\overline{H})^2} = kT^2 C_V.$$

56. $$\overline{f\frac{\partial H}{\partial q_i}} = -kTfe^{-\frac{f-H}{kT}}\Big|_{q_i=-\infty}^{q_i=+\infty} d\Gamma' + kT\int e^{-\frac{f-H}{kT}}\frac{\partial f}{\partial q_i}\,d\Gamma$$

$= kT\overline{\dfrac{\partial f}{\partial q_i}}$, where $d\Gamma' = dp_1 \ldots dp_N\,dq_1 \ldots dq_{i-1}dq_{i+1} \ldots \times$ $\times\, dq_N$. We prove the second equality in a similar way.

57. For small densities the partition function of a real gas is

$$Z = Z_{\text{perfect}}\left(1+\frac{N}{V}\frac{N}{2}\beta\right)$$

where $\beta = -8v_0 + \frac{8U_0 v_0}{kT}$, and $v_0 = \frac{1}{8}\frac{4\pi r_0^3}{3}$ is the volume of one molecule.

We denote $n = N/V$ and $b = 4Nv_0$, and we get

$$E = E_0 - \frac{nbU_0}{1+nb\left(\frac{U_0}{kT}-1\right)},$$

$$C_V = (C_V)_{\text{perfect}} - k\left(\frac{nbU_0}{kT}\right)^2 \frac{1}{\left[1+nb\left(\frac{U_0}{kT}-1\right)\right]^2}$$

As temperature falls, the heat capacity falls too.

58. The critical point is determined by three equations:

$$\left(p-\frac{a}{V^2}\right)(V-b) = RT$$

$$\left(\frac{\partial p}{\partial V}\right)_T = 0, \quad \frac{\partial^2 p}{\partial V^2} = 0$$

From this we find p_{cr}, T_{cr}, V_{cr} and get the sought relationship (the reduced van der Waals equation).

59. The partition function of the system is (see the solution to Problem 21)

$$Z = (z_i)^N$$

$$z_i = 4\pi V\,(4\pi^2 M\mu)^{3/2}\,(kT)^3 \int_0^\infty r^2 e^{-U/kT}\,dr$$

Considering the anharmonicity of atomic vibrations, we get

$$\begin{aligned} U(r-r_0) &= \frac{1}{2}\frac{\partial^2 U}{\partial r^2}\bigg|_{r=r_0}(r-r_0)^2 + \frac{1}{6}\frac{\partial^3 U}{\partial r^3}\bigg|_{r=r_0}(r-r_0)^3 \\ &\quad + \frac{1}{24}\frac{\partial^4 U}{dr^4}\bigg|_{r=r_0}(r-r_0)^4 + \dots \\ &= \gamma(r-r_0)^2 + \alpha(r-r_0)^3 + \beta(r-r_0)^4 + \dots \end{aligned}$$

where α, β, γ are the respective derivatives. If we consider the anharmonic addition small, we can calculate the integral

approximately by expanding the exponential function with the anharmonic terms in a power series

$$z_i = A(kT)^3 r_0^2 \int_{-\infty}^{\infty} e^{\frac{-\gamma x^2}{kT}} \left[1 - \frac{\alpha x^3}{kT} - \frac{\beta x^4}{kT} + \frac{\alpha^2}{2(kT)^2} x^6 + \dots\right] dx$$

$$= A r_0^2 \sqrt{\frac{\pi}{\gamma}} (kT)^{7/2} [1 + BkT]$$

$$\left(A \equiv 4\pi V (4\pi^2 M\mu)^{3/2}, \quad B = \frac{15}{8}\frac{\alpha^2}{\gamma^3} - \frac{3}{4}\frac{\beta}{\gamma^2}\right)$$

Whence,

$$E = \frac{7}{2} NkT + Nk^2T^2B$$

$$C_V = \frac{7}{2} Nk + 2Nk^2BT = (C_V)_{\text{perfect}} + C_V'$$

So the correction to the heat capacity is $C_V' = 2Nk^2TB$, i.e. proportional to temperature, and therefore significant at high temperatures.

60. The atoms in the molecule are at some mean distance r_0 from each other. We determine the coefficient of linear expansion as follows:

$$\alpha_l = \frac{\bar{x}}{r_0 T}$$

where $\bar{x}$ is the mean displacement of an atom from the equilibrium position. But

$$\bar{x} = \frac{\int_{-\infty}^{\infty} x e^{-U/kT} dx}{\int_{-\infty}^{\infty} e^{-U/kT} dx}$$

and

$$U(x) = \left(\frac{A}{r_0^{12}} - \frac{B}{r_0^6}\right) + \frac{3}{r_0^2}\left(\frac{26A}{r_0^{12}} - \frac{7B}{r_0^6}\right) x^2 - \frac{28}{r_0^3}\left(\frac{13A}{r_0^{12}} - \frac{2B}{r_0^6}\right) x^3$$

where $r_0 = \sqrt[6]{2B/A}$ is determined from the condition $\frac{\partial U}{\partial r} = 0$. Since at the minimum point $\left.\frac{\partial^2 U}{\partial r^2}\right|_{r=r_0} > 0$, the

coefficient $\frac{13A}{r_0^{12}}-\frac{2B}{r_0^6}>0$. Then

$$\bar{x}=\frac{\int\limits_{-\infty}^{\infty} e^{-\frac{\gamma x^2}{kT}}\left(x+\frac{\delta x^3}{kT}\right)dx}{\int\limits_{-\infty}^{\infty} e^{-\frac{\gamma x^2}{kT}}\left(1+\frac{\delta x^2}{kT}\right)dx}=\frac{3}{4}kT\frac{\delta}{\gamma^2}$$

$$\left[\delta=\frac{28}{r_0^3}\left(\frac{13A}{r_0^{12}}-\frac{2B}{r_0^6}\right),\quad \gamma=\frac{3}{r_0^2}\left(\frac{26A}{r_0^{12}}-\frac{7B}{r_0^6}\right)\right]$$

And we get

$$\alpha_l=\frac{\bar{x}}{r_0T}=\frac{3}{4}\frac{\delta}{\gamma^2}\frac{k}{r_0}.$$

61. $\Omega(E)\propto\exp\left[\frac{(n+1)\,\alpha^{\frac{1}{n+1}}}{kn^{\frac{1}{n+1}}}E^{\frac{n}{n+1}}\right].$

63. In accordance with the assumption we write $S=f(w)$. Let the system consist of two independent subsystems. Then according to the general properties of entropy and probability we get

$$S=S_1+S_2=f(w_1)+f(w_2)$$
$$f(w_1)+f(w_2)=f(w_1\cdot w_2)$$

Hence,

$$S=f(w)=\text{constant}\times\ln w$$

To determine the constant we must apply this relationship to a perfect gas. The constant proves equal to the Boltzmann constant k.

64. $dQ=dE+p\,dV-(d\mathbf{P}\cdot\mathbf{E})$.

65. Let the functional dependence of A on B and C be expressed by $f(A, B, C)=0$. Then

$$\left(\frac{\partial f}{\partial A}\right)_{B,C}dA+\left(\frac{\partial f}{\partial B}\right)_{A,C}dB+\left(\frac{\partial f}{\partial C}\right)_{A,B}dC=0$$

If A is constant, this becomes

$$\left(\frac{\partial f}{\partial B}\right)_{A,C}\left(\frac{\partial B}{\partial C}\right)_A=-\left(\frac{\partial f}{\partial C}\right)_{A,B}$$

i. e.

$$\left(\frac{\partial B}{\partial C}\right)_A = -\left(\frac{\partial f}{\partial C}\right)_{A,\,B} \Big/ \left(\frac{\partial f}{\partial B}\right)_{A,\,C}$$

Similarly

$$\left(\frac{\partial C}{\partial A}\right)_B = -\left(\frac{\partial f}{\partial A}\right)_{B,\,C} \Big/ \left(\frac{\partial f}{\partial C}\right)_{A,\,B}$$

$$\left(\frac{\partial A}{\partial B}\right)_C = -\left(\frac{\partial f}{\partial B}\right)_{A,\,C} \Big/ \left(\frac{\partial f}{\partial A}\right)_{B,\,C}$$

Multiplication of these three equations yields

$$\left(\frac{\partial A}{\partial B}\right)_C \left(\frac{\partial B}{\partial C}\right)_A \left(\frac{\partial C}{\partial A}\right)_B = -1$$

Interchange of A and C in the second of these equations yields

$$\left(\frac{\partial A}{\partial C}\right)_B = 1 \Big/ \left(\frac{\partial C}{\partial A}\right)_B.$$

66. Selecting $A = p$ and $B = V$, as in Problem 65, we get

$$\alpha = p\beta\gamma.$$

67. *Hint.* We determine p_{cr}, T_{cr}, V_{cr} as in Problem 58 and introduce the variables

$$\pi = \frac{p}{p_{\mathrm{cr}}}, \quad \tau = \frac{T}{T_{\mathrm{cr}}}, \quad \omega = \frac{V}{V_{\mathrm{cr}}}$$

For the Dieterici equation of state we have

$$\pi(2\omega - 1) = \tau e^{2\left(1 - \frac{1}{\pi\omega}\right)}$$

For the other case

$$\left(\pi + \frac{4}{\omega^{5/3}}\right)(4\omega - 1) = 5\tau.$$

68. $T_{\mathrm{B}} = \dfrac{a}{bR}$.

69. The speed of propagation of sound waves in a gas is

$$v = \sqrt{\left(\frac{\partial p}{\partial \rho}\right)_S}$$

Considering the relationship

$$\left(\frac{\partial p}{\partial V}\right)_S = \frac{C_p}{C_V}\left(\frac{\partial p}{\partial V}\right)_T$$

(which will be proved in Problem 75) and the relationship $V\rho = \mu$ (μ is the molecular weight), we get

$$v = V\sqrt{-\frac{C_p}{\mu C_V}\left(\frac{\partial p}{\partial V}\right)_T}$$

But for the van der Waals gas

$$\left(\frac{\partial p}{\partial V}\right)_T = \frac{2a}{V^3} - \frac{RT}{(V-b)^2}$$

Then

$$v = \sqrt{\frac{C_p}{\mu C_V}\left[\frac{RT}{(V-b)^2} - \frac{2a}{V^3}\right]} \approx \frac{V}{V-b}\sqrt{\frac{C_p}{C_V\mu}RT}$$

$$\approx \frac{V}{V-b}\, v_{\text{perfect}}.$$

70. $HM^{-\gamma} = \text{constant}$ and $\gamma \equiv \dfrac{C_H}{C_M}$.

71. $(V-b)^R \exp\left(-\int\limits_0^T \frac{C_V}{T}\, dT\right) = \text{constant}.$

72. $C_E - C_D = \dfrac{TE^2}{4\pi\varepsilon}\left(\dfrac{\partial \varepsilon}{\partial T}\right)^2 > 0$

Hint. Use the relationships

$$dQ = \left(\frac{\partial E}{\partial T}\right)_a dT + \left[\bar{A} + \left(\frac{\partial E}{\partial a}\right)_T\right] da$$

$$dA_0 = -\frac{1}{4\pi}(\mathbf{E}\cdot d\mathbf{D}), \quad \mathbf{D} = \varepsilon(T)\,\mathbf{E}.$$

73. By definition $C_V = T\left(\dfrac{\partial S}{\partial T}\right)_V$. Hence,

$$\left(\frac{\partial C_V}{\partial V}\right)_T = T\frac{\partial}{\partial V}\left[\left(\frac{\partial S}{\partial T}\right)_V\right]_T = T\left(\frac{\partial^2 p}{\partial T^2}\right)_V$$

For the van der Waals gas,

$$\frac{\partial^2 p}{\partial T^2} = 0$$

i.e. C_V does not depend on the volume.

74. For the thermodynamic engine working according to the cycle shown in Fig. 48,

$$\eta = 1 - \frac{1}{\varepsilon^{\gamma-1}}$$

where $\gamma = \frac{C_p}{C_V}$. For the one working according to the cycle in Fig. 49,

$$\eta = 1 - \frac{1}{\gamma\varepsilon^{\gamma-1}} \frac{\rho^{\gamma}-1}{\rho-1}$$

Hint. Use the adiabatic equation $TV^{\gamma-1}$ = constant and

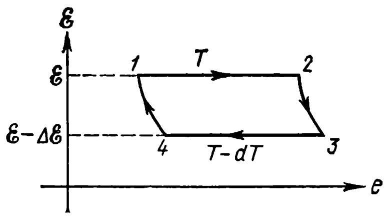

Fig. 69

the expression for the efficiency of a thermodynamic engine

$$\eta = \frac{A}{Q}$$

where A is the work done in the cycle, and Q is the heat liberated in the same cycle.

75. *Hint.* Use the properties of Jacobians [see formula (26) of Appendix 4] and the relationship (see the solution to Problem 80)

$$C_p - C_V = -T \frac{\left(\frac{\partial V}{\partial T}\right)_p^2}{\left(\frac{\partial V}{\partial p}\right)_T}.$$

76. We consider the reversible galvanic cell and we let it work in the Carnot cycle (Fig. 69). At first the cell works isothermically (1-2) with a constant e.m.f. $\mathscr{E}$, then it works adiabatically (2-3). We then send a current from an external source through it, and the work is done isothermically (3-4) with a constant e.m.f. $\mathscr{E} - \Delta\mathscr{E}$ and then adiabatically (4-1).

On isotherm 1-2 the cell is heated by an external heat source with $Q_1 = E_2 - E_1 + W_1$, where $W_1 = e\mathscr{E}$ is the work expended to transfer charges isothermically, and $(E_2 - E_1)$ is the change in the internal energy of the cell. That change is equal to the energy release $-qe$, where q is the energy release per passing charge. Thus

$$Q_1 = e\mathscr{E} - qe$$

On adiabat 2-3 the e.m.f. of the cell will diminish by $\Delta\mathscr{E}$ (and the temperature by ΔT). Therefore,

$$\eta = \frac{\Delta A}{Q_1} = \frac{e\Delta\mathscr{E}}{(\mathscr{E} - q)\, e}$$

where $\Delta A = e\Delta\mathscr{E}$ is the work done in the cycle equal to the area 1-2-3-4-1. But for the Carnot cycle

$$\eta = \frac{T - (T - \Delta T)}{T} = \frac{\Delta T}{T}$$

Solving the last two relationships simultaneously, we get the Gibbs-Helmholtz equation for a galvanic cell:

$$\mathscr{E} = q + T\left(\frac{\partial\mathscr{E}}{\partial T}\right)_V.$$

77. $\left(\frac{\partial T}{\partial V}\right)_E = \frac{\partial (T, E)}{\partial (T, V)} \cdot \frac{\partial (T, V)}{\partial (V, E)} = \frac{p - T\left(\frac{\partial p}{\partial T}\right)_V}{C_V}.$

78. The internal energy of the plasma is

$$E = E_{\text{perfect}} + E_1$$

where $E_{\text{perfect}} = C_V T$ is the average kinetic energy of the plasma (the internal energy of a perfect gas), $E_1 = \frac{1}{2} Ne\varphi_+ (0) - \frac{1}{2} Ne\varphi_- (0)$ is the average energy of the electrostatic interaction between the particles of the plasma, and $\varphi_+ (0)$ and $\varphi_- (0)$ are the potentials created by all the charges except the given one at its point of location. Let us determine these potentials. The charge density at a distance r from the fixed charge will be

$$\rho (r) = e (n_+ - n_-)$$

The potential φ is determined from the Poisson equation with consideration for the Boltzmann distribution:

$$\Delta\varphi = 4\pi e n_0 \left[\exp\left(\frac{e\varphi}{kT}\right) - \exp\left(-\frac{e\varphi}{kT}\right)\right]$$

where $n_0 = \frac{N}{V}$, and N is the number of particles of one kind in V.

If temperatures are high, $\frac{e\varphi}{kT} \ll 1$, and thus

$$\Delta\varphi = \varkappa^2\varphi \quad \left(\varkappa^2 = \frac{8\pi e^2 n_0}{kT}\right)$$

The solution of this equation has the form

$$\varphi(r) = \frac{C_1}{r} e^{-\varkappa r} + \frac{C_2}{r} e^{\varkappa r}$$

Here $\varphi(r)$ is the potential at a point that is at a distance r from the given charge, the potential being created by all the charges (including the given one). We require that the potential at infinity and at points where the charges are located be finite, and we get $C_1 = e$ and $C_2 = 0$. Hence,

$$\varphi_+(0) = -e\varkappa, \quad \varphi_-(0) = e\varkappa$$

$$E = E_{\text{perfect}} - Ne^2\sqrt{\frac{8\pi e^2 N}{VkT}}.$$

79. Using the general relationship $E = -T^2\frac{\partial}{\partial T}\left(\frac{F}{T}\right)_V$, we find

$$F = F_{\text{perfect}} - \frac{2}{3}Ne^2\sqrt{\frac{8\pi Ne^2}{VkT}}$$

Hence,

$$p = \frac{NkT}{V} - \frac{1}{3}Ne^2\sqrt{\frac{8\pi Ne^2}{kTV^3}}$$

$$S = S_{\text{perfect}} - \frac{1}{3}Ne^2\sqrt{\frac{8\pi Ne^2}{kT^3V}}$$

$$C_V = (C_V)_{\text{perfect}} + \frac{1}{2}Ne^2\sqrt{\frac{8\pi Ne^2}{3kT^3V}}.$$

80. (a) $$C_p = T\frac{\partial(S,p)}{\partial(T,V)}\cdot\frac{\partial(T,V)}{\partial(T,p)} = C_V - T\frac{\left(\frac{\partial p}{\partial T}\right)_V^2}{\left(\frac{\partial p}{\partial V}\right)_T};$$

(b) $$C_V = T\frac{\partial(S,V)}{\partial(p,T)}\cdot\frac{\partial(p,T)}{\partial(T,V)} = C_p + T\frac{\left(\frac{\partial V}{\partial T}\right)_p^2}{\left(\frac{\partial V}{\partial p}\right)_T}$$

[see formula (29), Appendix 4].

81. *Hint.* Use the first law of thermodynamics expressed in terms of p and T.

82. *Hint.* Use the results of Problem 64.

83. $$C_p - C_V = \frac{R}{1-\frac{2a(V-b)^2}{RTV^3}}.$$

84. $$Q = \frac{C_V}{R}\left[\left(p+\frac{a}{V_2^2}\right)(V_2-b) - \left(p+\frac{a}{V_1^2}\right)(V_1-b) + a\left(\frac{1}{V_1}-\frac{1}{V_2}\right)\right].$$

85. $pV^n =$ constant, where $n = \frac{C_p - C}{C_V - C}$ is the coefficient of polytropy.

86. $S = S_0 + C_p \ln T - \alpha V_0 p =$ constant.

87. $C_V \ln T + \alpha p_0 V =$ constant.

88. It follows from the first law of thermodynamics, $dQ = dE + p_0\,dV$, that in the first case $dQ_p = d(E + p_0 V)$, and in the second case $dQ_V = dE$. Therefore,

$$dQ_p - dQ_V = p_0\,dV$$

Hence,

$$Q_p - Q_V = RT(n_2 - n_1)$$

where n_1 and n_2 are the numbers of moles of the reacting particles before and after the reaction. For the reaction $H_2 + \frac{1}{2}O_2 \rightleftarrows H_2O$ we have $n_1 = 3/2$, $n_2 = 1$ and

$$Q_p - Q_V = -\frac{1}{2}RT.$$

89. $$\eta = \frac{T - T_0}{T}.$$

90. *Hint.* Use the second law of thermodynamics for reversible processes.

91. $(\Delta S)_p = \dfrac{C_p}{T\left(\dfrac{\partial V}{\partial T}\right)_p}\Delta V.$

92. Let the isotherm intersect the adiabat twice, at points A and C. If we examine the closed cycle ABC (Fig. 70), we find that $\oint p dV \neq 0$. On the other hand, the entropy

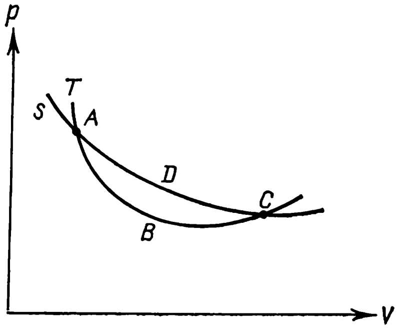

Fig. 70

of the system in states A and C is the same, i.e. $S_A = S_C$. Hence,

$$A = Q = \oint T\,dS = T\int_{ABC} dS = 0$$

The solution of this contradiction lies in the fact that an isotherm cannot intersect an adiabat twice. The principle that follows from this is that thermodynamic states resulting from an initial state isothermically cannot result from the same state adiabatically.

93. *Hint.* Use the first law of thermodynamics.

94. $C_p = C_V$ for $t = 4\,°\mathrm{C}$.

95. The change in the pressure of a perfect gas with altitude is determined by the relationship

$$dp = -\rho g\,dh = -\frac{g\mu}{R}\frac{p}{T}\,dh$$

where the molecular weight of air $\mu = 29$ kg/kmol, and $g = 9.8$ m/s^2. Since the process under consideration is adiabatic, we have

$$\frac{dp}{p} = \frac{\gamma}{\gamma - 1} \frac{dT}{T}$$

These two equations yield

$$\frac{dT}{dh} = -\frac{\gamma - 1}{\gamma} \frac{g\mu}{R} \left(\gamma = \frac{C_p}{C_V}\right)$$

Since $\gamma = 1.4$ and $R = 8.314 \times 10^3$ JK^{-1}kmol^{-1}, we get

$$\frac{dT}{dh} \approx -10\,\mathrm{Kkm}^{-1}$$

(The actual mean gradient is about —8 Kkm^{-1}.)

96. The condition for mechanical equilibrium found in Problem 95 holds when the temperature varies. Now, will this state of atmospheric air be stable if due to convection the temperatures of different layers become the same? To find the answer we will consider two volumes of gas of unit mass situated at heights h and $h + dh$. Let us assume these two volumes change places. If their total energy increases as a result, the earlier state of the system will be stable despite convection.

The change in the energy will be

$$\Delta E + p\Delta V$$

since both temperature and pressure change with altitude. Hence, the condition of stability will have the form

$$\frac{dE}{dh} + p\frac{dV}{dh} \geqslant 0$$

But

$$p\frac{dV}{dh} = \frac{R}{\mu}\frac{dT}{dh} - \frac{V\,dp}{dh}, \quad \frac{dp}{dh} = -\rho g$$

$$E = C_V T = \frac{pV}{\gamma - 1}$$

From this we get the condition of stability of atmospheric air in relation to convection:

$$\frac{dT}{dh} \geqslant -\frac{\mu g (\gamma - 1)}{R\gamma}.$$

97. $Q=\frac{E^2}{8\pi}T\frac{\partial\varepsilon}{\partial T}$.

99. Using the expression for F, we get

$$S=-\left(\frac{\partial F}{\partial T}\right)_V=S_0+\frac{VE^2}{8\pi}\left(\frac{\partial\varepsilon}{\partial T}\right)_V$$

But $\varepsilon=1+\frac{4\pi np_0^2}{3kT}$ (see the solution to Problem 49). Hence,

$$S=S_0-\frac{VE^2(\varepsilon-1)}{4\pi T}$$

where $S_0=-\left(\frac{\partial F_0}{\partial T}\right)_V$, and

$$E=F+TS=E_0-\frac{VE^2(\varepsilon-1)}{4\pi}$$

where $E_0=F_0+TS_0$.

100. For black-body radiation with an energy density u the radiation pressure is $p=u/3$. Now we apply to this radiation the relationship $T\,dS=dE+p\,dV$, which easily takes the form

$$T\left(\frac{\partial p}{\partial T}\right)_V=\left(\frac{\partial E}{\partial V}\right)_T+p$$

and we get

$$u=\sigma T^4$$

where σ is a universal constant that cannot be defined thermodynamically ($\sigma=7.56\times10^{-16}$ $\mathrm{JK^{-4}m^{-3}}$). All the other thermodynamic functions are now easily determined. Since $E=\sigma T^4V$, it follows that

$$S=\frac{4}{3}\sigma T^3V,\quad F=-\frac{1}{3}\sigma T^4V$$

$$\Phi=F+pV=0 \text{ and } H=TS$$

We find the heat capacities:

$$C_V=4\sigma T^3V$$

and since for black-body radiation an isobaric process is also an isothermic process ($p=\frac{1}{3}\sigma T^4$), we get

$$C_p=+\infty$$

which means that $\gamma=\frac{C_p}{C_V}=\infty$, although the adiabatic equation for black-body radiation is

$$pV^{4/3}=\text{constant}\quad (S \text{ is constant}).$$

101. Since

$$Z=\sum_{N=0}^{\infty}\frac{1}{N!\,h^{3N}}e^{\frac{\mu N}{kT}}\int e^{-\frac{H}{kT}}\,d\Gamma=\exp\left(e^{\frac{\mu}{kT}}\frac{V}{\lambda^3}\right)$$

where $\lambda=\frac{h}{\sqrt{3mkT}}$ is the "thermal" de Broglie wave, we have

$$\mu=kT\ln\frac{\bar{N}\lambda^3}{V},\quad p=\frac{\bar{N}\,kT}{V}$$

$$S=\frac{3}{2}\bar{N}k+\bar{N}k\ln\frac{eV}{\bar{N}\lambda^3}$$

(e is the base of natural logarithms).

102. The sought probability is

$$\rho_N=\frac{1}{N!}\left(\frac{V}{\lambda^3}\right)^N e^{\frac{\Xi+\mu\bar{N}}{kT}}$$

But since

$$\Xi=-kTe^{\mu/kT}\frac{V}{\lambda^3},\quad \bar{N}=e^{\frac{\mu}{kT}}\frac{V}{\lambda^3}$$

we have

$$\rho_N=\frac{1}{N!}e^{-\bar{N}}\bar{N}^N$$

which is the Poisson distribution.

103. The dependence of μ on U can be found from the relationship

$$\mu=\left(\frac{\partial E}{\partial N}\right)_{S,V}=\mu_0+U$$

Specifically, in a homogeneous gravitational field,

$$\mu=\mu_0+mgz.$$

104. If the average number of particles with spin "up" is $\bar{N}_1$, and with spin "down" is $\bar{N}_2$, we get (see the solution

to Problem 103)

$$\mu_1 = kT \ln \bar{N}_1 - \mu_B B + \varphi(T)$$
$$\mu_2 = kT \ln \bar{N}_2 + \mu_B B + \varphi(T)$$

where μ_B is the Bohr magneton. Now using the condition of equilibrium, $\mu_1 = \mu_2$, we have

$$\frac{\bar{N}_2}{\bar{N}_1} = \exp\left(-\frac{2\mu_B B}{kT}\right).$$

105. $pV = -\Xi$.

106. *Hint.* Use the relationships

$$\Xi = -pV$$
$$dE = TdS - pdV + \mu\, dN$$
$$dH = TdS + Vdp + \mu\, dN.$$

107. $$C_V = kT\left[\left(\frac{\partial S}{\partial T}\right)_\mu - \frac{\left(\frac{\partial N}{\partial T}\right)_\mu^2}{\left(\frac{\partial N}{\partial \mu}\right)_T}\right].$$

108. The Clausius-Clapeyron equation is

$$\frac{dp}{dT} = \frac{\lambda}{T(V_2 - V_1)}$$

Far from the critical point we can assume that $V_2 \gg V_1$ (V_1 and V_2 are the molar volumes of the vapour and solid phases, respectively), and the gas obeys the equation of state for perfect gas. Hence,

$$p = \text{constant} \times e^{-\frac{\lambda}{RT}}.$$

109. Let the concentration of the solution be $c = \frac{n}{N} \ll 1$ and the Gibbs free energy of the solvent be $\Phi = N\mu_0(p, T)$. We denote by β the change in Φ when one molecule of the solute is added to the solvent. Then by virtue of the condition $c \ll 1$ we can assume that the molecules of the solute do not interact. Now if we take into account the indistinguishability of the molecules of the solute, we get the following expression for the Gibbs free energy:

$$\Phi = N\mu_0 + n\beta + kT \ln n! = N\mu_0 + nkT \ln\left(\frac{n}{e} e^{\frac{\beta}{kT}}\right)$$

since $\ln n! \approx n \ln \frac{n}{e}$. Next we consider that Φ must be a homogeneous function of degree one in n and N. This yields

$$e^{\frac{\beta}{kT}-1} = \frac{f(p, T)}{N}$$

(so that under the logarithm sign is a function of degree zero in n and N), and

$$\Phi = N\mu_0 + nkT \ln \frac{n}{N} + n\varphi(p, T)$$

where $\varphi(p, T) = kT \ln f(p, t)$. Using the obtained expression for Φ, we find the respective chemical potentials of the solution and the solute:

$$\mu = \frac{\partial \Phi}{\partial N} = \mu_0 - kTc, \quad \mu_1 = \frac{\partial \Phi}{\partial n} = kT \ln c + kT + \varphi(p, T).$$

110. For the solute and the solvent the conditions of equilibrium in the gravitational field have the form (T = constant)

$$kT \ln c + \varphi(p, T) = -mgz + \text{constant}$$

$$\mu_0 - kTc + Mgz = \text{constant}$$

If we differentiate these equations with respect to z and bear in mind that the volume of the solution is

$$V = \left(\frac{\partial \Phi}{\partial p}\right)_T = N\left(\frac{\partial \mu_0}{\partial p}\right)_T + n\left(\frac{\partial \varphi}{\partial p}\right)_T$$

we have

$$\frac{1}{c}\frac{dc}{dz} + \frac{V_1}{V_0}\frac{dc}{dz} = \frac{g}{kT}\left(M\frac{V_1}{V_0} - m\right)$$

where M is the mass of a molecule of the solvent, m the mass of a molecule of the solute, and $V_0 = \left(\frac{\partial \mu_0}{\partial p}\right)_T$ and $V_1 = \left(\frac{\partial \varphi}{\partial p}\right)_T = \frac{V - NV_0}{n}$ are the volumes related to one molecule of the solvent and the solute, respectively. In the first-order approximation in c we get the solution

$$c = c_0 \exp\left[-\frac{gz}{kT}\left(m - M\frac{V_1}{V_0}\right)\right]$$

This is the barometric height formula corrected for Archimedes' principle.

111. It is easily shown that for black-body radiation the Gibbs free energy $\Phi = E - TS + pV = 0$ (see the solution to Problem 100). Hence, $\mu = 0$.

112. Consider a system whose state is described by the variables a_1 and a_2; and A_1 and A_2 are the generalized forces. If $\varphi(a_1, a_2)$ is a certain function of the state of the system, we have

$$\left(\frac{\partial A_1}{\partial a_2}\right)_{a_1} = \left(\frac{\partial A_2}{\partial a_1}\right)_{a_2}$$

and

$$\left(\frac{\partial a_1}{\partial A_2}\right)_{A_1} = \left(\frac{\partial a_2}{\partial A_1}\right)_{A_2}$$

because $f = \varphi - a_1A_1 - a_2A_2$ will also be a function of state. Using these relationships, we find that

$$\left(\frac{\partial a_1}{\partial A_1}\right)_{a_2} = \frac{\partial(a_1, a_2)}{\partial(A_1, a_2)} = \left(\frac{\partial a_1}{\partial A_1}\right)_{A_2} - \left(\frac{\partial a_1}{\partial A_2}\right)^2_{A_1} \left(\frac{\partial A_2}{\partial a_2}\right)_{A_1}$$

This yields

$$\left(\frac{\partial a_1}{\partial A_1}\right)_{a_2} < \left(\frac{\partial a_1}{\partial A_1}\right)_{A_2}$$

since $\left(\frac{\partial A_2}{\partial a_2}\right)_{A_1} > 0$ because of the condition of stability. This inequality expresses the following physical fact: an external force A_1 changes the parameter a_1 and, hence, the parameters a_2 and A_2, and the measure of this effect will be the derivative $\frac{\partial a_1}{\partial A_1}$. At first, obviously, there will be almost no change in A_2. The force exerted will be characterized by $\left(\frac{\partial a_1}{\partial A_1}\right)_{A_2}$. But later new values of a_2 will establish themselves in the system and the external force exerted will be characterized by $\left(\frac{\partial a_1}{\partial A_1}\right)_{a_2}$, which proves to be reduced.

113. The condition for mechanical equilibrium of the "vapour-liquid" system in the presence of a surface separating the two phases is of the form (μ and p are the same for both phases)

$$dF = -p_1\,dV_1 - p_2\,dV_2 + \sigma\,dS$$

where p_1 is the pressure in the drop, p_2 is the pressure in the vapour, σ is called the surface tension, and dS is the surface differential. Keeping in mind the constancy of the total volume $V_1 + V_2$ and the sphericity of the interface, we get

$$p_1 = p_2 + \frac{2\sigma}{R}.$$

114. Since $\sum\limits_i \nu_i = 0$ in this reaction,

$$K = \frac{m_2 m_3}{m_1 m_4}$$

i.e. the affinity constant does not depend on p.

115. Assume that a drop of liquid of radius R has formed in the vapour. The Helmholtz free energy will then change by

$$\Delta F = (\mu_2 - \mu_1) N + \sigma S$$

where N is the number of particles in the drop, and μ_1 and μ_2 are the chemical potentials of the vapour and the liquid, respectively. But

$$N = \frac{4\pi R^3}{3v}$$

where v is the volume related to one particle in the liquid.

If $\mu_2 < \mu_1$, it is easy to prove that ΔF reaches its maximum at point $R_{cr} = \frac{2\sigma v}{\mu_1 - \mu_2}$. From this it follows that if the "nucleus" has $R > R_{cr}$, the drop will grow. Otherwise, it will vaporize.

116. Assume a drop of radius R takes on an ion with charge q and radius R_0. The change in the Helmholtz free energy will be

$$\Delta F = \frac{4\pi R^3}{3v}(\mu_2 - \mu_1) + 4\pi\sigma R^2 + \frac{\varepsilon}{8\pi}\int_{R_0}^{R} \frac{q^2}{\varepsilon r^2}\, dr$$

$$+ \frac{1}{8\pi}\int_{R}^{\infty} \frac{q^2}{r^2}\, dr - \frac{1}{8\pi}\int_{R_0}^{\infty} \frac{q^2}{r^2}\, dr$$

$$= \frac{4\pi R^3}{3v}(\mu_2 - \mu_1) + 4\pi\sigma R^2 + \frac{q^2}{2}\left(1 - \frac{1}{\varepsilon}\right)\left(\frac{1}{R} - \frac{1}{R_0}\right)$$

where ε is the permittivity of the liquid. The last term in this expression is always negative and grows as R grows, i.e.

ΔF decreases. Hence, a drop can grow even if $\mu_2 > \mu_1$, i.e. even in vapour that has not yet reached the saturation point.

117. Bearing in mind the results of Problem 101 and applying the condition of chemical equilibrium, we get

$$\mu_A = \mu_I + \mu_e$$

$$kT \ln \frac{c_A p h^3}{(2\pi m_A kT)^{3/2}\, kT} + \mu_A^0 = kT \ln \frac{c_I p h^3}{(2\pi m_A kT)^{3/2}\, kT}$$

$$+ \mu_I^0 + kT \ln \frac{c_e p h^3}{(2\pi m_e kT)^{3/2}\, kT} + \mu_e^0$$

We denote the initial number of atoms as N and introduce the degree of single ionization α. We then get the following relationships for the number of particles of the reactants and the respective concentrations:

$$n_e = \alpha N, \qquad n_I = \alpha N, \qquad n_A = (1-\alpha) N,$$

$$c_e = \frac{\alpha}{1+\alpha}, \quad c_I = \frac{\alpha}{1+\alpha}, \quad c_A = \frac{1-\alpha}{1+\alpha}$$

From this we get

$$\alpha = \left[1 + \frac{p}{(kT)^{5/2}} \left(\frac{h^2}{2\pi m_e}\right)^{3/2} e^{\frac{\varepsilon_0}{kT}}\right]^{-1/2}.$$

118. When p and T are constant, the first law of thermodynamics yields

$$\Delta Q = \Delta E + p_0 \Delta V = -T^2 \frac{\partial}{\partial T} \frac{\Delta \Phi}{T}$$

where Φ is the Gibbs free energy. But the change in the chemical potential for a reversible chemical reaction with constant p and T is expressed as

$$\Delta \Phi = \sum_i \mu_i \Delta N_i = \sum_i \mu_i \nu_i = kT \ln K(T)$$

Hence,

$$\Delta Q = -kT^2 \frac{\partial}{\partial T} \ln K(T)$$

119. For the reaction $H_2 \rightleftarrows H + H$ we have

$$\ln K(T) = \frac{2\chi'(T) - \chi(T)}{kT}$$

where $\chi(T) = \mu_{H_2} - kT \ln p_{H_2}$ and $\chi'(T) = \mu_H - kT \ln p_H$.

The chemical potential μ_H was determined in Problem 101, and μ_{H_2} we find from the grand partition function for a diatomic perfect gas:

$$Z = \sum_{N=0}^{\infty} \frac{1}{N!} e^{\mu N/(kT)} z^N_{\text{transl}} z^N_{\text{rot}} z^N_{\text{vibr}}$$

$$= \sum_{N=0}^{\infty} \frac{1}{N!} \left[e^{\mu/(kT)} V \left(\frac{2\pi M kT}{h^2} \right)^{3/2} \frac{1}{1-e^{\frac{h\nu}{kT}}} \left(\frac{8\pi^2 IkT}{\gamma h^2} \right) \right]^N$$

where M is the reduced mass of the two atoms in a molecule, and I is the moment of inertia of a molecule. The "zero point" for the energy of vibrations $\varepsilon_n = h\nu \left(n + \frac{1}{2}\right)$ is the energy of zero-point vibrations, and γ is a constant that takes account of the symmetry of a hydrogen molecule (for a diatomic gas $\gamma = 2$, the two-fold degeneracy).
But

$$\sum_{N=0}^{\infty} \frac{x^N}{N!} = e^x \quad \text{and} \quad pV = kT \ln Z$$

then

$$p = e^{\mu/kT} \left(\frac{2\pi M}{h^2} \right) (kT)^{7/2} \frac{4\pi^2 I}{h^2} - \frac{1}{1-e^{h\nu/kT}}$$

$$\mu = kT \ln p - \frac{7}{2} kT \ln kT - kT \ln \left[\left(\frac{2\pi M}{h^2} \right)^{3/2} \frac{4\pi^2 I}{h^2} \right]$$

$$+ kT \ln \frac{1}{1-e^{h\nu/kT}} + \mu^0$$

Hence, for $kT \gg h\nu$, we have

$$\ln K(T) = -\frac{kT}{2} + \frac{2\mu^0_{\mathrm{H}} - 2\mu^0_{\mathrm{H}_2}}{kT} + \ln \frac{4I\sqrt{\pi}}{\nu m^{3/2}}$$

(m is the mass of a hydrogen atom), and

$$K(T) = \frac{4I\sqrt{\pi}}{(kT)^{3/2}\nu} \frac{1}{\sqrt{kT}} e^{\Delta\varepsilon/kT} \quad \text{and} \quad \Delta Q = \frac{kT}{2} + \Delta\varepsilon$$

where $\Delta\varepsilon = 2\mu^0_{\mathrm{H}} - 2\mu^0_{\mathrm{H}_2}$ is the dissociation energy for a hydrogen molecule.

120. $\Delta p = \frac{(N_2 - N_1)}{V} kT.$

121. We consider one quantum mechanical state to be a thermodynamic system and calculate the function Ξ_i:

$$\Xi_{i\binom{\text{Fermi}}{\text{Bose}}} = -kT \ln \sum_{n_i \neq 0} e^{\frac{n_i(\mu-\varepsilon_i)}{kT}} = \binom{-1}{+1} kT \ln\left(1 \pm e^{\frac{\mu-\varepsilon_i}{kT}}\right) \quad (1)$$

(The energy of such a "system" is $n_i\,\varepsilon_i$, n_i is the number of particles in the "system", summation is done from 0 to 1 for fermions and from 0 to ∞ for bosons, and μ is negative for bosons.)

The number of states with a given momentum (see Problem 125 from the section "Quantum Mechanics") integrated over all possible directions of the momentum is

$$dg(p) = g_s \frac{4\pi p^2\,dp}{h^3} V$$

Hence the number of states with a given energy is

$$dg(\varepsilon) = BV\sqrt{\varepsilon}\,d\varepsilon$$

Since the spectrum is dense, the total thermodynamic function Ξ is

$$\Xi_{\binom{\text{Fermi}}{\text{Bose}}} = \sum_i \Xi_{i\binom{\text{Fermi}}{\text{Bose}}} = \binom{-1}{+1} BV\,kT \int_0^\infty \ln\left(1 \pm e^{\frac{\mu-\varepsilon}{kT}}\right)\sqrt{\varepsilon}\,d\varepsilon$$

Integrating by parts, we get

$$\Xi_{\binom{\text{Fermi}}{\text{Bose}}} = -\frac{2}{3} BV \int_0^\infty \frac{e^{\frac{\mu-\varepsilon}{kT}}\varepsilon^{3/2}}{1 \pm e^{\frac{\mu-\varepsilon}{kT}}}\,d\varepsilon$$

$$= -\frac{2}{3}\int_0^\infty \frac{\varepsilon\,dg(\varepsilon)}{e^{\frac{\varepsilon-\mu}{kT}} \pm 1} = -\frac{2}{3}E.$$

122. $S = k\left[\ln\frac{\varepsilon}{\varepsilon-E} + \frac{E}{\varepsilon}\ln\left(\frac{\varepsilon-E}{E}\frac{g_2}{g_1}\right)\right]$.

This relationship is depicted in Fig. 71, from which we can see that in the energy range from $\frac{\varepsilon g_2}{g_1+g_2}$ to ε the derivative $\left(\frac{\partial S}{\partial E}\right)a_s$, which by definition is equal to $1/T$, is nega-

tive, i.e. in this energy range the temperature must be considered negative. Note that negative temperatures correspond to higher energies than positive. We will always meet this

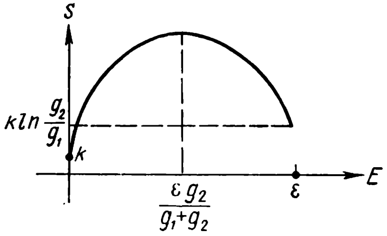

Fig. 71

fact when we have quasi-equilibrium systems with an energy spectrum with an upper limit.

123. $C_V = \frac{Nk}{2}\left(\frac{\hbar\omega}{kT}\right)^2 \frac{1}{\sinh^2 \frac{\hbar\omega}{2kT}}$.

124. $E = \frac{\varepsilon}{e^{\frac{\varepsilon}{kT}}-1} - \frac{n\varepsilon}{e^{\frac{n\varepsilon}{kT}}-1}$.

125. The average number of phonons with an energy $h\nu$ is $\bar{n} = \frac{1}{e^{\frac{h\nu}{kT}}-1}$. The number of states in the frequency range $[\nu, \nu+d\nu]$ is

$$\Omega(\nu)\, d\nu = 4\pi\nu^2\, d\nu\, \frac{3V}{c^3}$$

where

$$\frac{3}{c^3} = \frac{2}{c_t^3} + \frac{2}{c_l^3}$$

Hence,

$$E = \frac{12V}{c^3}\pi\left(\frac{kT}{h}\right)^4 h \int_0^{\frac{h\nu_{\max}}{kT}} \frac{x^3\, dx}{e^x - 1}$$

where $\nu_{\max}$ is the maximal Debye frequency determined from the condition $\int\limits_0^{\nu_{\max}} \Omega(\nu)\, d\nu = 3N$ (conservation of numbers of degrees of freedom of the system).

If temperatures are low, we can set the upper limit at infinity. Then [see formula (21), Appendix 4]

$$E = \frac{4}{5}\frac{V}{c^3}\frac{k^4T^4}{h^3}\pi^5.$$

126. $C_V = N(3\pi^2)^{2/3}\frac{1}{3c\hbar}\left(\frac{V}{N}\right)^{1/3} kT.$

127. $j_x = 2en\left(\frac{m}{h}\right)^3 \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} dv_y\, dv_z \frac{kT}{m} \ln\left[1 + e^{\frac{\mu - eW}{kT}} e^{-\frac{mv^2}{2kT}}\right].$

The factor $e^{-\frac{W-\mu}{kT}}$ for metals is about e^{-10} at $T \approx 10^2$ K. So using the fact that for $x \ll 1$ we have $\ln(1+x) \approx x$, we get

$$j_x = \frac{4\pi e n m}{h^3}(kT)^2 e^{-\frac{w}{kT}}$$

Here $w = W - \mu$ is the effective work function. This is the quantum Richardson equation.

128. $p = \frac{1}{5}(3\pi^2)^{2/3}\frac{\hbar^2}{m}\left(\frac{N}{V}\right)^{5/3}.$

129. $p = \frac{2}{3}\frac{E}{V} = \frac{2}{5}\frac{N}{V}\mu_0\left[1 + \frac{5\pi^2}{12}\left(\frac{kT}{\mu_0}\right)^2\right]$, where $\mu_0 = \frac{\hbar^2}{2m}\left(3\pi^2\frac{N}{V}\right)^{2/3}$ is the Fermi level for electrons in a metal at $T = 0$.

130. $$S = \begin{cases} -k\sum\limits_i [\overline{n_i}\ln\overline{n_i} + (1-\overline{n_i})\ln(1-\overline{n_i})] & \text{for fermions} \\ -k\sum\limits_i [\overline{n_i'}\ln\overline{n_i'} - (1+\overline{n_i'})\ln(1+\overline{n_i'})] & \text{for bosons} \end{cases}$$

Here

$$\overline{n_i} = \frac{1}{e^{\frac{\varepsilon_i - \mu}{kT}} + 1} \quad \text{and} \quad \overline{n_i'} = \frac{1}{e^{\frac{\varepsilon_i - \mu}{kT}} - 1}.$$

131. $C_V = \left(\frac{\partial E}{\partial T}\right)_V$, $E = \int\limits_0^\infty \frac{\varepsilon g(\varepsilon)\, d\varepsilon}{\exp\left(\frac{\varepsilon - \mu}{kT}\right) + 1}$, where μ is found from the condition that

$$\int\limits_0^\infty \frac{g(\varepsilon)\, d\varepsilon}{\exp\left(\frac{\varepsilon - \mu}{kT}\right) + 1} = N$$

At low temperatures these integrals can be represented as

$$N = \int\limits_0^\mu g(\varepsilon)\, d\varepsilon + \frac{\pi^2}{6} k^2 T^2 g'(\mu) + \ldots$$

$$E = \int\limits_0^\mu \varepsilon g(\varepsilon)\, d\varepsilon + \frac{\pi^2}{6} k^2 T^2 g(\mu) + \frac{\pi^2}{6} k^2 T^2 \mu g'(\mu) + \ldots$$

[see formula (20) of Appendix 4]. But for $T = 0$

$$\int\limits_0^{\mu_0} g(\varepsilon)\, d\varepsilon = N$$

and also $\mu - \mu_0$ is much less than μ_0 and μ as T tends to zero. Therefore, with an accuracy up to terms quadratic in T we find that

$$\int\limits_{\mu_0}^{\mu} g(\varepsilon)\, d\varepsilon + \frac{\pi^2}{6} k^2 T^2 g'(\mu) + \ldots = 0$$

$$g(\mu_0)(\mu - \mu_0) + \frac{\pi^2}{6} k^2 T^2 g'(\mu_0) = 0$$

$$\mu = \mu_0 - \frac{\pi^2}{6}(kT)^2 \left[\frac{d}{d\varepsilon} \ln g(\varepsilon)\right]_{\varepsilon = \mu_0}$$

Using the obtained value of μ, we get with the same degree of accuracy

$$E = \int\limits_0^{\mu_0} \varepsilon g(\varepsilon)\, d\varepsilon + (\mu - \mu_0)\mu_0 g(\mu_0) + \frac{\pi^2}{6}(kT)^2 g(\mu_0)$$

$$+ \frac{\pi^2}{6} k^2 T^2 \mu_0 g'(\mu_0) + \ldots = E_0 + \frac{\pi^2}{6}(kT)^2 g(\mu_0)$$

Hence,

$$C_V \approx \frac{\pi^2}{3} k^2 g(\mu_0) T.$$

132. Let us show that $\overline{dA}_{\text{rev}} = -\sum_i \bar{n}_i \ d\varepsilon_i$. By definition, for reversible processes $\bar{n}_i = -\frac{\partial \Xi_i}{\partial \mu}$. If we use formula (1) from the solution of Problem 121 and note that $\frac{\partial \Xi_i}{\partial \mu} = -\frac{\partial \Xi_i}{\partial \varepsilon_i}$, and $d\varepsilon_i = \sum_{k=1}^{l} \frac{\partial \varepsilon_i}{\partial a_k} da_k$, we construct

$$\sum_i \bar{n}_i \, d\varepsilon_i = \sum_i \frac{\partial \Xi_i}{\partial \varepsilon_i} \sum_{k=1}^{l} \frac{\partial \varepsilon_i}{\partial a_k} da_k = \sum_{k=1}^{l} \frac{\partial \left(\sum_i \Xi_i\right)}{\partial a_k} da_k = -\overline{dA}_{\text{rev}}$$

Thus,

$$dQ = dE + \overline{dA}_{\text{rev}} = \sum_i \varepsilon_i \, d\bar{n}_i.$$

133. $$C_V = \left(\frac{\partial E}{\partial T}\right)_V = \frac{\partial}{\partial T}\left[\frac{V\hbar}{4\pi^2 A^{3/2}} \left(\frac{k}{\hbar}\right)^{5/2} T^{5/2} \int_0^\infty \frac{x^{3/2}\,dx}{e^x - 1}\right]$$

$$= \frac{15}{32} \pi \frac{V\hbar}{A^{3/2}} \left(\frac{k}{\pi\hbar}\right)^{5/2} \times 1.341 T^{3/2} = BT^{3/2}$$ [see formula (21) of Appendix 4].

134. The state of a solid whose atoms are considered to be harmonic oscillators is determined by the volume V and the set of oscillators ($n_i = 1, 2, \ldots$). In such an approximation the energy of the crystal is

$$E = E_0(V) + \sum_{i=1}^{3N-6} \left(\bar{n}_i + \frac{1}{2}\right) h\nu_i(V)$$

where $E_0(V)$ is the interaction energy of N motionless particles of the crystal.

Knowing E, we can find the pressure $p = kT\left(\frac{\partial \ln Z}{\partial V}\right)_T$, where

$$Z = e^{-E_0(V)/kT} \prod_{i=1}^{3N-6} \frac{1}{2\sinh \frac{h\nu_i}{kT}}$$

Then

$$p = -\left(\frac{\partial E_0}{\partial V}\right)_T - \frac{1}{V} \sum_{i=1}^{3N-6} \left(\frac{1}{2} h\nu_i + \frac{h\nu_i}{e^{h\nu_i/(kT)} - 1}\right)\left(\frac{\partial \ln \nu_i}{\partial \ln V}\right)$$

$$= -\frac{\partial E_0}{\partial V} + \frac{\gamma E}{V}$$

Hence,

$$\left(\frac{\partial p}{\partial T}\right)_V = \frac{\gamma C_V}{V}$$

Now, using the formulas in the solution to Problem 65 and the definitions for α_l and β, we find the sought relationship.

135. $\Omega(E) \propto e^{S/k} \propto e^{\frac{4}{5}\pi^4 N\left(\frac{T}{T_0}\right)^3}$, where $T_0 = h\nu_{\max}/k$.

136. Let the lower edge of the conduction energy band be E_c, and the higher edge E_v. Then from the condition of the electroneutrality of a semiconductor ($n = p$) for the simplest law of dispersion,

$$\varepsilon_n = E_c + \frac{\hbar^2 k^2}{2m_n}, \quad \varepsilon_p = E_v - \frac{\hbar^2 k^2}{2m_p}$$

we get

$$2\left(\frac{m_n kT}{2\pi\hbar^2}\right)^{3/2} e^{\frac{\mu - E_c}{kT}} = 2\left(\frac{m_p kT}{2\pi\hbar^2}\right)^{3/2} e^{\frac{E_v - \mu}{kT}}$$

Hence,

$$\mu = \frac{E_g^0 - \xi T}{2} + \frac{3}{4} kT \ln \frac{m_n}{m_p}.$$

137. $n = \frac{\sqrt{\pi}}{\pi^2 \hbar^3}(8m_t^2 m_l)^{3/2}(kT)^{3/2} F_{1/2}\left[\frac{1}{kT}(\mu - E_c)\right]$ [see formula (22) of Appendix 4]. When integrating with respect to $\mathbf{k}$, we took infinite limits, which, obviously, is

possible only if these limits lie outside the area of occupied states. Practically speaking, this is always the case.

138. The total concentration n is equal to the sum of the concentrations in the lower (n_1) and upper (n_2) bands, i.e.

$$n = n_1 + n_2 = \frac{2}{(2\pi)^3} \int d\mathbf{k} \left\{1 + \exp\left[\frac{1}{kT}(E_1 - \mu)\right]\right\}^{-1} + \frac{2}{(2\pi)^3} \int d\mathbf{k} \left\{1 + \exp\left[\frac{1}{kT}(E_2 - \mu)\right]\right\}^{-1}$$

Here $E_1(\mathbf{k}) = \frac{\hbar^2 |\mathbf{k}|^2}{2m_1}$ and $E_2(\mathbf{k}) = \frac{\hbar^2 |\mathbf{k}|^2}{2m_2} + E_0$. The energy is reckoned from the lower edge of the first band. Then

$$n = 2\left(\frac{m_1 kT}{2\pi\hbar^2}\right)^{3/2} F_{1/2}\left(\frac{\mu}{kT}\right) + 2\left(\frac{m_2 kT}{2\pi\hbar^2}\right)^{3/2} F_{1/2}\left(\frac{\mu - E_0}{kT}\right)$$

Using the asymptotic form of the Fermi integral [see formulas (23) and (24) of Appendix 4], we find

$$\mu = kT \ln \frac{n}{2\left(\frac{kT}{2\pi\hbar^2}\right)^{3/2} \left[m_1^{3/2} + m_2^{3/2} e^{-E_0/(kT)}\right]}$$

$$n = \frac{1}{3\pi^2}\left(\frac{2m_1}{\hbar^2}\right)^{3/2} \mu^{3/2}\left[1 + \left(\frac{m_2}{m_1}\right)^{3/2}\left(1 - \frac{E_0}{\mu}\right)^{3/2} \Theta(\mu - E_0)\right]$$

where the step function

$$\Theta(x) = \begin{cases} 1 & \text{for } x > 0 \\ 0 & \text{for } x < 0. \end{cases}$$

139. $n = \frac{2}{(2\pi)^3} \int f(E)\, d\mathbf{k} = \int\limits_0^\infty g(E) f(E)\, dE$, where $g(E) = \frac{k^2}{\pi^2} \frac{dk}{dE}$ is the density of states. When the energy is small, we easily find that

$$k^2 = \frac{1}{2\gamma}\left(1 - \sqrt{1 - \frac{8m\gamma E}{\hbar^2}}\right) \approx \frac{2mE}{\hbar^2}\left(1 + \frac{2m\gamma E}{\hbar^2}\right)$$

Hence,

$$n = 2\left(\frac{m_n kT}{2\lambda\hbar^2}\right)^{3/2}\left[F_{1/2}\left(\frac{\mu}{kT}\right) + \frac{15}{2} \frac{m_n \gamma kT}{\hbar^2} F_{3/2}\left(\frac{\mu}{kT}\right)\right].$$

140. If we choose the edge of the conduction band as the energy's zero point, we get

$$g(E)=\frac{1}{2\pi^2}\left[\frac{2m(0)}{\hbar^2}\right]^{3/2}\left[E^{1/2}\left(1+\frac{E}{E_g}\right)^{1/2}\left(1+\frac{2E}{E_g}\right)\right]$$

In the case of a degenerate semiconductor we have

$$n=\frac{1}{3\pi^2}\left[\frac{2m(0)}{\hbar^2}\right]^{3/2}\mu^{3/2}\left(1+\frac{\mu}{E_g}\right)^{3/2}.$$

141. Let the lower edge of the conduction band *1* (Fig. 72) be the zero point for calculating the energy. The probability that the quantum state with energy ε is not occupied by an electron, i.e. is a hole by definition, is expressed as

$$f'(\varepsilon)=1-f(\varepsilon)=\frac{1}{\exp\left(\mu-\frac{\varepsilon}{kT}\right)+1}$$

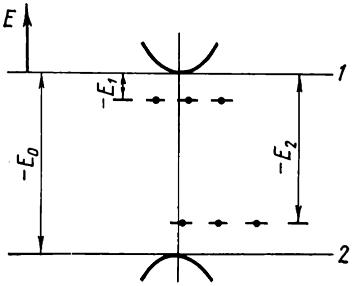

Fig. 72

The energy of an electron in the conduction band is $\varepsilon_n=\frac{\hbar^2k^2}{2m_n}$, on the donor level it is $\varepsilon=-E_1$, on the acceptor level $\varepsilon=-E_2$, and in the valence band *2* it is $\varepsilon=-E_0-\varepsilon_p$, where $\varepsilon_p=\frac{\hbar^2k^2}{2m_p}$ is the kinetic energy of a hole. Now we write the condition for the electroneutrality of the semiconductor:

$$\int\limits_{\text{band } 1}\frac{2g(\varepsilon)\,d\varepsilon}{\exp\left(\frac{\varepsilon_n-\mu}{kT}\right)+1}+\sum_{\text{accep}}\frac{1}{\exp\left(\frac{-E_2-\mu}{kT}\right)+1}$$
$$=\int\limits_{\text{band } 2}\frac{2g'(\varepsilon)\,d\varepsilon}{\exp\left(\frac{\mu+E_0+\varepsilon_p}{kT}\right)+1}+\sum_{\text{donor}}\frac{1}{\exp\left(\frac{\mu-E_1}{kT}\right)+1}$$

Assuming that the electrons in the conduction band *1* and the holes in the valence band *2* obey Boltzmann's statistics,

we get for $x = \exp(\mu/kT)$ the following equation:

$$\frac{(2\pi m_n kT)^{3/2}}{4\pi^3\hbar^3}x+\frac{n_2}{e^{-E_2/kT}\dfrac{1}{x}+1}$$

$$=\frac{(2\pi m_p kT)^{3/2}}{4\pi^3\hbar^3}e^{-E_0/kT}\frac{1}{x}+\frac{n_1}{e^{-E_1/kT}x+1}$$

where n_1 and n_2 are, respectively, the concentrations of donors and acceptors.

Obviously, the concentrations of electrons in the conduction band n and holes in the valence band p are expressed as follows:

$$n=\frac{(2\pi m_n kT)^{3/2}}{4\pi^3\hbar^3}e^{\mu/(kT)}$$

$$p=\frac{(2\pi m_p kT)^{3/2}}{4\pi^3\hbar^3}e^{-\frac{E_0-\mu}{kT}}$$

This is an equation of the fourth degree in x. Therefore, we consider its solution in a special case: $n_2 = 0$ for $kT \gg E_1$. Then

$$\mu=kT\ln\frac{4\pi^3\hbar^3 n_1}{(2\pi m_n kT)^{3/2}}$$

and $n \approx n_1$, i.e. all the donors are ionized.

We can examine an acceptor-type semiconductor in a similar way.

142. *Hint.* Use the results of Problem 141.

143. The energy of electrons and holes in 1 cm^3 is

$$E=2\int_0^\infty \varepsilon f(\varepsilon)\,g(\varepsilon)\,d\varepsilon+2\int_0^\infty (E_0-\varepsilon')\,f'(\varepsilon')\,g'(\varepsilon')\,d\varepsilon'$$

$$=n\,(3kT+E_0)$$

where $f(\varepsilon)$ and $g(\varepsilon)$ are the distribution function and the density of states of the electrons, $f'(\varepsilon')$ and $g'(\varepsilon')$ are the same for holes (see Problem 141), and

$$n=\frac{(2\pi\sqrt{m_n m_p}\,kT)^{3/2}}{4\pi^2\hbar^3}e^{-E_0/kT}$$

Hence

$$C_V = kn\left[\frac{15}{2}+3\left(\frac{E_0}{kT}\right)+\frac{1}{2}\left(\frac{E_0}{kT}\right)^2\right]$$

which is valid for $E_0 \gg kT$ (the electron gas is nondegenerate).

144. $p=\frac{c}{8\pi^2\hbar^3}\left[p_0\left(\frac{2}{3}p_0^2-m^2c^2\right)\sqrt{p_0^2+m^2c^2}+(mc)^4\times\right.$ $\left.\times \operatorname{arc\,sinh}\frac{p_0}{mc}\right]$, where $p_0=(3\pi^2)^{1/3}\hbar\left(\frac{N}{V}\right)^{1/3}$ is the maximum Fermi momentum.

145. $\frac{N_{\text{ortho}}}{N_{\text{para}}}=3\frac{\sum\limits_{j=1,\,3\ldots}(2j+1)\,e^{-\frac{T_c}{T}j(j+1)}}{\sum\limits_{j=0,\,2\ldots}(2j+1)\,e^{-\frac{T_c}{T}j(j+1)}}$, where $T_c=$ $=\frac{\hbar^2}{2Ik}$. Keeping in mind that $T\ll T_c$, we get

$$\frac{N_{\text{ortho}}}{N_{\text{para}}}=9e^{-2\frac{T_c}{T}}.$$

146. Let $W(E_k)$ be the probability that a particle is in a state with energy E_k. Clearly, if two particles with energies E_1 and E_2, respectively, are to pass into states with the energies E_3 and E_4 after colliding, the latter must not be occupied (here the Pauli exclusion principle comes in).

Now using the hypothesis that for a large number of particles the probabilities of the direct and reverse processes are the same, we find for the process

$$E_1+E_2 \rightleftarrows E_3+E_4$$

the following functional relationship:

$$W(E_1)\,W(E_2)\,[1-W(E_3)]\,[1-W(E_4)]$$
$$=W(E_3)\,W(E_4)\,[1-W(E_1)]\,[1-W(E_2)]$$

We introduce the function $f(E)=W^{-1}(E)-1$ and get

$$f(E_3)\,f(E_4)=f(E_1)\,f(E_2)$$

The solution of this equation for $E_1 + E_2 = E_3 + E_4$ is given by the function

$$f(E) = Ae^{\alpha E}$$

Hence

$$W(E) = \frac{1}{Ae^{\alpha E}+1}.$$

147. Consider the crystal as an aggregate of $3N$ normal modes (more precisely, $3N - 6$, but this is unimportant when N is great) with frequencies ω_j. The average energy associated with each mode is $\left(n_j + \frac{1}{2}\right) \hbar\omega_j$. The energy related to the jth oscillator will be

$$NM\omega_j^2\overline{r_j^2} = \left(\bar{n}_j + \frac{1}{2}\right) \hbar\omega_j$$

where M is the mass of an atom, and r_j is the contribution of the jth normal mode to the atom's displacement.

We divide the equality by $NM\omega_j^2$ and sum up with respect to j, and we get

$$\overline{r^2} = \frac{\hbar}{MN} \sum_j \frac{\left(\bar{n}_j + \frac{1}{2}\right)}{\omega_j}$$

We replace the sum by an integral and, keeping in mind the Debye nature of the spectrum, we find that

$$\overline{r^2} = \frac{\hbar}{2MN} \int_0^{\omega_{\max}} \coth\left(\frac{\hbar\omega}{2kT}\right) \frac{g(\omega)}{\omega}\, d\omega$$

where $g(\omega) = \frac{9Nw^2}{\omega_{\max}^3}$ (see the solution to Problem 125); $\omega_{\max}$ is determined from the condition that the number of degrees of freedom of the system must be conserved: $\int_0^{\omega_{\max}} g(\omega)\, d\omega = 3N.$

If we define the Debye temperature as $T_D = \frac{\hbar\omega_{\max}}{k}$, then for $T \ll T_D$ we have

$$\overline{r^2} = \frac{9\hbar^2}{2MkT_D}\left(1 + \frac{2\pi^2 T^2}{3T_0}\right).$$

148. $\rho' = \rho(1-\beta^2)^{-1}$, $T' = \frac{\partial E'}{\partial S'} = T\sqrt{1-\beta^2}$.

149. $u(\nu, T) = \frac{8\pi n^3(\nu)}{c^3} h\nu^3 \frac{d\ln[\nu n(\nu)]}{d\ln\nu} \frac{h\nu}{e^{h\nu/kT}-1}$.

150. The energy of an electron with a spin magnetic moment μ_B in a magnetic field B (directed along the r-axis) is

$$\varepsilon = \frac{p^2}{2m} \pm \mu_B B$$

Hence, the magnetization (magnetic moment density) is

$$M = \mu_B \int_0^\infty [g(\varepsilon + \mu_B B) - g(\varepsilon - \mu_B B)] f(\varepsilon)\, d\varepsilon$$

$$\approx 2\mu_B^2 B \int_0^\infty \frac{dg}{d\varepsilon} \frac{1}{e^{\frac{\varepsilon-\mu}{kT}}+1}\, d\varepsilon$$

Now we determine the magnetic susceptibility in certain special cases:

(a) for $T = 0$ K

$$\chi = \frac{M}{B} = 2\mu_B^2 g(\mu_{T=0})$$

where $\mu_{T=0}$ is the chemical potential of the electrons at absolute zero;

(b) strong degeneracy ($kT \ll \mu$)

$$\chi = 2\mu_B^2 g(\mu)\left[1 + \frac{\pi^2}{6} k^2T^2 \frac{\partial^2 \ln g(\mu)}{\partial\mu^2} + \dots\right]$$

[see formula (20) of Appendix 4];

(c) no degeneracy ($kT \gg \mu$)

$$\chi = \frac{n\mu_B^2}{kT}$$

since $f = e^{\frac{\mu-\varepsilon}{kT}}$ (n is the number of particles per unit volume).

151. In the case of Boltzmann's statistics the partition function of the system is $Z = \frac{1}{N!} z_i^N$. If the electron gas is in a cube with edge L and the field B is directed along the

z-axis, we have

$$z_i = \frac{L}{h} \int_{-\infty}^{\infty} e^{-\frac{p_z^2}{2m_n kT}} dp_z \sum_l \Omega(E_l) e^{-\frac{2\mu^* B\left(l+\frac{1}{2}\right)}{kT}} 2\cosh\frac{\mu_B B}{kT}$$

where $\mu^* = \frac{e\hbar}{2m_n c}$, $\mu_B = \frac{e\hbar}{2mc}$, and $\Omega(E_l)$ is the degeneracy multiplicity of the lth energy level, which is

$$\Omega(E_l) = \frac{L^2}{h^2} \int\int dp_x\, dp_y = \frac{L^2 eB}{h^2 c}$$

where the integration region is

$$2\mu^* Bl < \frac{p_x^2 + p_y^2}{2m_n} < 2\mu^* B\,(l+1)$$

since all the levels that at $B = 0$ lie in the integration region combine for $B \neq 0$ into one level $2\mu^* B\left(l + \frac{1}{2}\right)$.

In the case of weak fields

$$M = N_A \mu_B \tanh\frac{\mu_B B}{kT} - N_A \mu_B \coth\left(\frac{\mu^* B}{kT} - \frac{kT}{\mu^* B}\right)$$
$$\approx \frac{N_A B}{kT}\left(\mu_B^2 - \frac{1}{3}\mu^{*2}\right)$$

$$\chi \approx \frac{N_A}{kT}\left(\mu_B^2 + \frac{1}{3}\mu^{*2}\right).$$

152. $\chi = -\frac{2\pi\,(2m)^{3/2}}{3h^3}\mu_B^2.$

Hint. Determine the Helmholtz free energy using the Euler-MacLaurin summation formula

$$\sum_{n=0}^{\infty} f(n) = \int_0^{\infty} f(n)\,dn + \frac{1}{2} f(0) - \frac{1}{12} f'(0) + \frac{1}{720} f'''(0) - \dots .$$

153. $\Delta = \sqrt{\frac{kT}{\alpha}}.$

154. $\overline{\Delta p^2} = -kT\left(\frac{\partial p}{\partial V}\right)_s.$

155. $\Delta S = -\frac{mgl\varphi^2}{T}.$

156. The solution is similar to that of Problem 154. In the variables V and T we have

$$\overline{\Delta p\,\Delta T}=k^2T^2\frac{1}{C_V}\left(\frac{\partial p}{\partial T}\right)_V,\quad \overline{\Delta V\,\Delta S}=kT\left(\frac{\partial V}{\partial T}\right)_p.$$

157. The relationships in the condition of the problem follow from the solution of Problem 154.

158. $\overline{(\Delta H)^2}=\dfrac{g_1g_2a^2}{(g_2+g_1e^{a/kT})^2}e^{a/kT}$, where $a=\varepsilon_2-\varepsilon_1$.

159. Using the relationship $\overline{\Delta N^2}=kT\left(\dfrac{\partial\overline{N}}{\partial\mu}\right)_{T,\,a_s}$, we get the following results:

(a) $\overline{\Delta N^2}=\overline{N},\quad \delta=\dfrac{1}{\sqrt{\overline{N}}}$;

(b) $\overline{\Delta n_i^2}=\overline{n}_i\,(1-\overline{n}_i),\quad \delta=\sqrt{\dfrac{1-\overline{n}_i}{\overline{n}_i}}$;

(c) $\overline{\Delta n_i^2}=\overline{n}_i\,(1+\overline{n}_i),\quad \delta=\sqrt{\dfrac{1+\overline{n}_i}{\overline{n}_i}}$.

For the Fermi gas (b), as we can see, the fluctuation of the number of particles becomes zero for $\overline{n}_i=0$ and 1, though the relative fluctuation is equal to ∞ for $\overline{n}_i=0$. For the Bose gas the fluctuation remains finite (equal to unity) even for very great $\overline{n}_i$.

160. *Hint.* Use the following relationships:

$$\left(\frac{\partial\overline{N}}{\partial T}\right)_{\mu/T}=\left(\frac{\partial\overline{N}}{\partial\mu}\right)_T\left[\frac{\mu}{T}-\left(\frac{\partial\mu}{\partial T}\right)_{\overline{N}}\right]$$

$$\left(\frac{\partial E}{\partial\overline{N}}\right)_T=\mu-T\left(\frac{\partial\mu}{\partial T}\right)_{\overline{N}}.$$

161. $\overline{(\Delta r_c)^2}=\dfrac{\overline{\rho(\mathbf{r})}}{N^2}\displaystyle\int r^2\,dx\,dy\,dz=\dfrac{3}{5N}R_0^2$, where R_0 is the radius of the sphere, and $\rho(\mathbf{r})$ is the density of the gas.

162. By virtue of the homogeneity of time and the reversibility of the equations of mechanics we have

$$\overline{q_i^tq_k^0}=\overline{q_i^0q_k^t},\quad \overline{q_i^0q_k^0}=\overline{q_i^tq_k^t}$$

Now we find

$$\overline{(q_i^t - q_i^0)(q_k^t - q_k^0)} = \overline{q_i^t (q_k^t - q_k^0)} - \overline{q_i^0 (q_k^t - q_k^0)} = \overline{2q_i^t (q_k^t - q_k^0)}$$

But the mean of a physical quantity $F = F(p, q)$ over a nonequilibrium ensemble $H_0(p, q) - \alpha Q$ will be

$$\overline{F} = \int F(p, q, \alpha) e^{\frac{\Psi - H_0 + \alpha Q - \alpha Q_0}{kT}} d\Gamma$$

where α is an additional force switched on at $t = 0$ in the direction of coordinate Q.

After finding the derivative of this expression with respect to α and making α vanish, we get

$$\left(\frac{\overline{\partial F^t}}{\partial \alpha} - \frac{\overline{\partial F}}{\partial \alpha}\right)_{\alpha=0} = \frac{1}{kT} \overline{F^t (Q^t - Q^0)}$$

Hence,

$$\overline{(q_i^t - q_i^0)(q_k^t - q_k^0)} = 2kT \left(\frac{\overline{\partial q_i^t}}{\partial \alpha_k}\right)_{\alpha_k=0}$$

In the special case, for $q_i = q_k = q$ and $\alpha_i = \alpha_k = \alpha$,

$$\overline{(q^t - q^0)^2} = 2kT \left(\frac{\overline{\partial q^t}}{\partial \alpha}\right)_{\alpha=0}.$$

163. Let us assume that the equation of motion for a Brownian particle can be represented in the form

$$m\ddot{x} = -6\pi a\eta\dot{x} + \alpha$$

Here the first term in the right-hand side is the drag force (the Stokes flow), and the second term is an additional force switched on at $t = 0$. Solving this equation with the initial conditions

$$\dot{x}\mid_{t=0} = 0, \qquad x\mid_{t=0} = 0$$

we get

$$x(t) = \frac{\alpha}{6\pi a\eta} t + \frac{\alpha m}{(6\pi a\eta)^2} \left(e^{-\frac{6\pi a\eta}{m} t} - 1\right)$$

For $t \gg \frac{m}{6\pi a\eta}$,

$$\overline{\Delta x^2} = 2Dt$$

which is known as the Einstein diffusion equation $\left(\text{here } D=\frac{kT}{6\pi a\eta}\right)$.

164. $N_A=\frac{RTt}{3\pi a\eta\overline{(\Delta x^2)}}$.

165. If the z-axis is directed along the gravitational field, we get

$$\overline{(z^t-z^0)^2}=2Dt+\left(\frac{mg}{6\pi a\eta}\right)^2 t^2.$$

166. The equation of diffusion in the presence of an external field $U=U(x)$ is of the form

$$\frac{\partial n}{\partial t}=D\frac{\partial}{\partial x}\left[\frac{n}{kT}\frac{\partial U}{\partial x}+\frac{\partial n}{\partial x}\right]$$

where D is the diffusion coefficient, and $n(x)$ is the concentration of particles.

For a stationary process $n=n(x)$,

$$\frac{\partial j_x}{\partial x}=0$$

where the flow of particles along the x-axis is

$$j_x=-D\left(\frac{\partial U}{\partial x}\frac{n}{kT}+\frac{\partial n}{\partial x}\right)=-De^{-U/kT}\frac{\partial}{\partial x}\left(ne^{U/kT}\right)$$

We integrate this expression from x_1 to x_2 and get

$$j_x=\frac{n(x_2)\exp\left[\frac{U(x_2)}{kT}\right]-n(x_1)\exp\left[\frac{U(x_1)}{kT}\right]}{\int\limits_{x_1}^{x_2}\exp\left[\frac{U(x)}{kT}\right]dx}(-D).$$

167. $\sigma_{\alpha\beta}=\delta_{\alpha\beta}\frac{ne^2}{m_n}\tau(\mu)$, where

$$n=\left(\frac{2m_n\mu}{3\pi^2h^3}\right)^{3/2},\quad \delta_{\alpha\beta}=\begin{cases}1 & \text{for } \alpha=\beta\\ 0 & \text{for } \alpha\neq\beta\end{cases}$$

We made no assumption in this case about the dependence of τ on velocity.

168. For this problem we must determine the current density j_1 and the heat flux j_2 along the x-axis:

$$j_1 = \int e v_x f \, d\mathbf{v}$$

$$j_2 = \int \frac{mv^2}{2} v_x f \, d\mathbf{v}$$

The distribution function f is found from the kinetic equation

$$f = f_0 - \tau \left(\frac{eE}{m} \frac{\partial f_0}{\partial v_x} + v_x \frac{\partial f_0}{\partial x} \right)$$

We assume that

$$f_0 = n \left(\frac{m}{2\pi kT} \right)^{3/2} e^{-\frac{mv^2}{2kT}}$$

and that the field E and the temperature gradient $\frac{\partial T}{\partial x}$ have little effect on f_0. Then

$$f = f_0 + \frac{\tau e E}{kT} v_x f_0 - \frac{\tau v_x}{kT^2} \left[\varepsilon - \frac{3}{2} kT \right] f_0 \frac{\partial T}{\partial x}$$

Hence [see formula (3) of Appendix 4],

$$j_1 = \frac{4enA}{3m\sqrt{\pi}} \Gamma\left(\frac{l+5}{2} \right) \left(\frac{2kT}{m} \right)^{l/2} \left[eE - \left(\frac{l}{2} + 1 \right) k \frac{\partial T}{\partial x} \right]$$

$$j_2 = \frac{4nA}{3m\sqrt{\pi}} \Gamma\left(\frac{l+7}{2} \right) \left(\frac{2kT}{m} \right)^{l/2} kT \left[eE - \left(\frac{l}{2} + 2 \right) k \frac{\partial T}{\partial x} \right]$$

Now we represent j_1 and j_2 in the form

$$j_1 = L_{11} E + k L_{12} \frac{\partial T}{\partial x}$$

$$j_2 = L_{21} E + k L_{22} \frac{\partial T}{\partial x}$$

It follows from these relationships that the electrical and thermal conductivities are

$$\sigma = \frac{4}{3} \frac{e^2 n A}{m\sqrt{\pi}} \Gamma\left(\frac{l+5}{2} \right) \left(\frac{2kT}{m} \right)^{l/2} \quad \left(\frac{\partial T}{\partial x} = 0 \right)$$

$$K = \left[-L_{22} + \frac{L_{12} L_{21}}{L_{11}} \right] = \frac{3nA}{3m\sqrt{\pi}} \Gamma\left(\frac{l+7}{2} \right) \left(\frac{2kT}{m} \right)^{l/2} kT$$

(the thermal conductivity is measured at $j_1 = 0$).

We consider two special cases.

(1) $l = 0$ (the relaxation time $\tau = A$ = constant). Then

$$\sigma = \frac{ne^2 A}{m}, \quad K = \frac{5}{2} n\lambda_0 \sqrt{\frac{kT}{2\pi m}}$$

(2) $\tau = \frac{\lambda_0}{v}$ (λ_0 is the mean free path of an electron). Then

$$\sigma = \frac{4}{3} e^2 n\lambda_0 \sqrt{\frac{1}{2\pi mkT}}, \quad K = \frac{8}{3} n\lambda_0 \sqrt{\frac{kT}{2\pi m}}.$$

169. For E and H constant, the stationary Boltzmann equation is

$$-e\left(\mathbf{E} + \left[\frac{\mathbf{v}}{c} \times \mathbf{H}\right]\right)\frac{\partial f}{\partial \mathbf{p}} = -\frac{f - f_0}{\tau}$$

If $\varepsilon(p) = \frac{p^2}{2m}$, the term with $\mathbf{H}$ will vanish when we substitute f_0 for f in the right-hand side. Hence,

$$-e(\mathbf{v}\cdot\mathbf{E})\frac{\partial f_0}{\partial \varepsilon} - \frac{e}{c}[\mathbf{v}\times\mathbf{H}]\frac{\partial(f - f_0)}{\partial \mathbf{p}} = -\frac{f - f_0}{\tau}$$

We look for the solution of this equation in the form

$$f = f_0 - (\mathbf{v}\cdot\mathbf{a})\frac{\partial f_0}{\partial \varepsilon}$$

where vector $\mathbf{a}(\varepsilon)$ is unknown.

Substituting the assumed f into Boltzmann's equation, we find the following equation for $\mathbf{a}(\varepsilon)$:

$$-e(\mathbf{v}\cdot\mathbf{E}) + [\mathbf{v}\times\boldsymbol{\omega}]\,\mathbf{a} = +\frac{(\mathbf{v}\cdot\mathbf{a})}{\tau} \quad \left(\boldsymbol{\omega} = \frac{e\mathbf{H}}{mc}\right)$$

Hence,

$$-e\mathbf{E} + [\boldsymbol{\omega}\times\mathbf{a}] = \frac{\mathbf{a}}{\tau}$$

We first find the scalar product of this equation and $\boldsymbol{\omega}$, and then the vector product of the equation and $\boldsymbol{\omega}$. From the two products we find $\mathbf{a}$:

$$\mathbf{a} = -\frac{e\tau}{1 + \omega^2\tau^2}\{\mathbf{E} + \tau^2(\boldsymbol{\omega}\cdot\mathbf{E})\,\boldsymbol{\omega} + \tau[\boldsymbol{\omega}\times\mathbf{E}]\}$$

Hence,

$$j_\alpha = -e^2 \int_0^\infty \left(-\frac{\partial f_0}{\partial \varepsilon}\right)\frac{\tau v^2}{3(\omega^2\tau^2 + 1)}[\delta_{\alpha\beta} + \tau^2\omega_\alpha\omega_\beta \pm (1 - \delta_{\alpha\beta})\,\tau\omega_\gamma]\,E_\gamma\,\frac{2dp}{h^3}$$

where $\delta_{\alpha\beta}=\begin{cases} 1 & \text{for} \quad \alpha=\beta, \\ 0 & \text{for} \quad \alpha\neq\beta \end{cases}$ and $\omega=\dfrac{e\,|\mathbf{H}|}{mc}$.

In the last expression we use "plus" or "minus" depending on whether the permutation of α, β, γ is even or odd with respect to x, y, z.

If the z-axis is directed along the field, we have

$$\sigma_{xx}=\sigma_{yy}=\frac{ne^2}{m_n}\frac{\tau(\mu)}{1+\omega^2\tau(\mu)}, \qquad \sigma_{zz}=\frac{ne^2}{m_n}\tau(\mu)$$

$$\sigma_{yx}=-\sigma_{xy}=\frac{ne^2}{m_n}\frac{\omega\tau^2(\mu)}{1+\omega^2\tau^2(\mu)}, \quad \sigma_{xz}=\sigma_{yz}=\sigma_{zy}=\sigma_{zx}=0.$$

170. The Boltzmann equation has the form

$$(\mathbf{v}\cdot\nabla_{\mathbf{r}})\,f(\mathbf{r},\ \mathbf{v})=-\frac{f-f_0}{\tau}$$

where

$$f_0=n\left(\frac{m}{2\pi kT}\right)^{3/2}\exp\left\{-\frac{m}{2kT}\left[(v_x-\beta y)^2+v_y^2+v_z^2\right]\right\}$$

since the field of velocities for the given flow is $v_x=\beta y$, $v_y=0$, $v_z=0$

Hence,

$$f=f_0-\tau(\mathbf{v}\cdot\nabla_{\mathbf{r}})\,f_0=f_0+\tau\beta v'_x\frac{\partial}{\partial v'_x}f_0(v'_x,\ v'_y,\ v'_z)$$

where $v'_x=v_x-\beta y$, $v'_y=v_y$, $v'_z=v_z$ are the velocities of molecules in relation to the moving gas. Now we determine the momentum carried along the y-axis through a perpendicular unit area in unit time (this is the shear stress X_y)

$$X_y=m\tau\beta\int d\mathbf{v}'\,v_y'^2 f_0=n\tau\,kT\frac{\partial v_x}{\partial y}=\eta\frac{\partial v_x}{\partial y}$$

The factor $\eta=nkT\tau$ is the coefficient of viscosity.

171. The function $f(\mathbf{r},\ \mathbf{v},\ t)$ satisfies the equation

$$\frac{\partial f}{\partial t}+(\mathbf{v}\cdot\nabla_{\mathbf{r}})\,f=0$$

At $t=0$,

$$f(\mathbf{r}_0,\ \mathbf{v}_0,\ t)=\rho_0(\mathbf{r}_0)\,f_0(\mathbf{v})$$

Since the particles move by inertia, $\mathbf{r}_0 = \mathbf{r} - \mathbf{v}t$, and

$$f(\mathbf{r}, \mathbf{v}, t) = \rho_0(\mathbf{r} - \mathbf{v}t) f_0(\mathbf{v})$$

will be the solution to Boltzmann's equation.

Hence,

$$\rho(\mathbf{r}, t) = \int f(\mathbf{r}, \mathbf{v}, t)\, d\mathbf{v}$$

$$= \left(\frac{m}{2\pi kTt^2}\right)^{1/2} \frac{1}{r} \int_0^\infty \left[e^{-\frac{m}{2kTt^2}(\mathbf{r}_0 - \mathbf{r})^2} - e^{-\frac{m}{2kTt^2}(\mathbf{r}_0 - \mathbf{r})^2}\right] \rho(r_0)\, r_0\, dr_0$$

Now taking into account that

$$\rho(r_0) = \begin{cases} \rho_0 & \text{for } 0 \leqslant r_0 \leqslant a \\ 0 & \text{for } r_0 > a \end{cases}$$

we find that

$$\rho(\mathbf{r}, t) = \frac{\rho_0}{2} \left\{ \frac{1}{r\sqrt{\frac{\pi m}{2kTt^2}}} \left[e^{-\frac{m}{2kTt^2}(r+a)^2} - e^{-\frac{m}{2kTt^2}(r-a)^2}\right] + \Phi\left[\sqrt{\frac{m}{2kTt^2}}(r+a)\right] - \Phi\left[\sqrt{\frac{m}{2kTt^2}}(r-a)\right] \right\}$$

where $\Phi(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt$.

Appendices

1. Basic formulas of vector analysis

$$\operatorname{grad} \varphi \equiv \nabla\varphi = \frac{\partial\varphi}{\partial x}\mathbf{i} + \frac{\partial\varphi}{\partial y}\mathbf{j} + \frac{\partial\varphi}{\partial z}\mathbf{k} \tag{1}$$

$$\int \varphi\mathbf{n}\, dS = \int \operatorname{grad}\varphi\, dV \quad (\mathbf{n} \text{ is an outward normal}) \tag{2}$$

$$\operatorname{div}\mathbf{a} \equiv (\nabla\cdot\mathbf{a}) = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z} \tag{3}$$

$$\int (\mathbf{a}\cdot\mathbf{n})\, dS = \int \operatorname{div}\mathbf{a}\, dV \tag{4}$$

$$\operatorname{curl}\mathbf{a} \equiv [\nabla\times\mathbf{a}] = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_x & a_y & a_z \end{vmatrix}$$

$$= \left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z}\right)\mathbf{i} + \left(\frac{\partial a_x}{\partial z} - \frac{\partial a_z}{\partial x}\right)\mathbf{j} + \left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\right)\mathbf{k} \tag{5}$$

$$\int [\mathbf{n}\times\mathbf{a}]\, dS = \int \operatorname{curl}\mathbf{a}\, dV \tag{6}$$

$$(\mathbf{a}\cdot\nabla)\,\mathbf{b} = a_x\frac{\partial\mathbf{b}}{\partial x} + a_y\frac{\partial\mathbf{b}}{\partial y} + a_z\frac{\partial\mathbf{b}}{\partial z} \tag{7}$$

$$\operatorname{div}\operatorname{grad}\varphi = \Delta\varphi = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2} \tag{8}$$

$$\Delta\mathbf{a} = \frac{\partial^2\mathbf{a}}{\partial x^2} + \frac{\partial^2\mathbf{a}}{\partial y^2} + \frac{\partial^2\mathbf{a}}{\partial z^2} \tag{9}$$

$$\operatorname{curl}\ \operatorname{grad}\ \varphi = 0 \tag{10}$$

$$\operatorname{div}\ \operatorname{curl}\mathbf{a} = 0 \tag{11}$$

$$\operatorname{curl}\ \operatorname{curl}\mathbf{a} = \operatorname{grad}\ \operatorname{div}\mathbf{a} - \Delta\mathbf{a} \tag{12}$$

$$\operatorname{grad}\ (\varphi f) = \varphi\ \operatorname{grad}\ f + f\ \operatorname{grad}\varphi \tag{13}$$

$$\operatorname{div}(\varphi\mathbf{a}) = \varphi \operatorname{div}\mathbf{a} + (\mathbf{a}\cdot\operatorname{grad}\varphi) \tag{14}$$

$$\operatorname{curl}(\varphi\mathbf{a}) = \varphi \operatorname{curl}\mathbf{a} + [\operatorname{grad}\varphi \times \mathbf{a}] \tag{15}$$

$$\operatorname{div}[\mathbf{a}\times\mathbf{b}] = \mathbf{b}\operatorname{curl}\mathbf{a} - \mathbf{a}\operatorname{curl}\mathbf{b} \tag{16}$$

$$\operatorname{grad}(\mathbf{a}\cdot\mathbf{b}) = (\mathbf{a}\cdot\nabla)\mathbf{b} + (\mathbf{b}\cdot\nabla)\mathbf{a} + [\mathbf{a}\times\operatorname{curl}\mathbf{b}] + [\mathbf{b}\times\operatorname{curl}\mathbf{a}] \tag{17}$$

$$\frac{1}{2}\operatorname{grad}a^2 = (\mathbf{a}\cdot\nabla)\mathbf{a} + [\mathbf{a}\times\operatorname{curl}\mathbf{a}] \tag{18}$$

$$\operatorname{curl}[\mathbf{a}\times\mathbf{b}] = (\mathbf{b}\cdot\nabla)\mathbf{a} - (\mathbf{a}\cdot\nabla)\mathbf{b}$$ 70

$$+\,\mathbf{a}\operatorname{div}\mathbf{b} - \mathbf{b}\operatorname{div}\mathbf{a} \tag{19}$$

$$\oint_L a_l\,dl = \int_S \operatorname{curl}_n\mathbf{a}\,dS \tag{20}$$

$$\oint_L \varphi\,dl = \int_S [\mathbf{n}\times\operatorname{grad}\varphi]\,dS \tag{21}$$

2. Curvilinear coordinates

Many problems are solved more easily if instead of Cartesian coordinates we use coordinates more naturally related to the problems. For a problem with axial symmetry, for example, it is convenient to use cylindrical coordinates; for a problem with spherical symmetry, spherical coordinates; etc. Such coordinate systems are called curvilinear coordinate systems (or, simply, curvilinear coordinates).

Since vectors and operations on vectors (div, curl, etc.) are usually defined in the Cartesian coordinate system, we must have formulas that express these operations in an arbitrary coordinate system.

Let us consider the projections of a vector in a curvilinear coordinate system; in this system the Cartesian coordinates of the vector, x, y, z, are functions of the curvilinear coordinates:

$$\begin{aligned} x &= x(q_1, q_2, q_3) \\ y &= y(q_1, q_2, q_3) \\ z &= z(q_1, q_2, q_3) \end{aligned} \tag{1}$$

or in vector form

$$\mathbf{r} = x(q_1, q_2, q_3)\,\mathbf{i} + y(q_1, q_2, q_3)\,\mathbf{j} + z(q_1, q_2, q_3)\,\mathbf{k}$$

The derivatives

$$\begin{aligned}
\frac{\partial \mathbf{r}}{\partial q_1} &= \frac{\partial x}{\partial q_1}\mathbf{i} + \frac{\partial y}{\partial q_1}\mathbf{j} + \frac{\partial z}{\partial q_1}\mathbf{k} \\
\frac{\partial \mathbf{r}}{\partial q_2} &= \frac{\partial x}{\partial q_2}\mathbf{i} + \frac{\partial y}{\partial q_2}\mathbf{j} + \frac{\partial z}{\partial q_2}\mathbf{k} \\
\frac{\partial \mathbf{r}}{\partial q_3} &= \frac{\partial x}{\partial q_3}\mathbf{i} + \frac{\partial y}{\partial q_3}\mathbf{j} + \frac{\partial z}{\partial q_3}\mathbf{k}
\end{aligned} \tag{2}$$

in the general case form a set of three linearly independent vectors, since the Jacobian of transformation (1) is not zero.

The absolute values of these vectors are, respectively,

$$\begin{aligned}
H_1 &= \sqrt{\left(\frac{\partial x}{\partial q_1}\right)^2 + \left(\frac{\partial y}{\partial q_1}\right)^2 + \left(\frac{\partial z}{\partial q_1}\right)^2} \\
H_2 &= \sqrt{\left(\frac{\partial x}{\partial q_2}\right)^2 + \left(\frac{\partial y}{\partial q_2}\right)^2 + \left(\frac{\partial z}{\partial q_2}\right)^2} \\
H_3 &= \sqrt{\left(\frac{\partial x}{\partial q_3}\right)^2 + \left(\frac{\partial y}{\partial q_3}\right)^2 + \left(\frac{\partial z}{\partial q_3}\right)^2}
\end{aligned} \tag{3}$$

They are called the Lamé parameters (also, scale factors).

If the three vectors (2) are divided by (3), i.e.

$$\mathbf{e}_1 = \frac{1}{H_1}\frac{\partial \mathbf{r}}{\partial q_1}, \quad \mathbf{e}_2 = \frac{1}{H_2}\frac{\partial \mathbf{r}}{\partial q_2}, \quad \mathbf{e}_3 = \frac{1}{H_3}\frac{\partial \mathbf{r}}{\partial q_3} \tag{4}$$

we get three unit vectors, which can be considered as the basis. The basis is not, in general, orthogonal, but from now on we shall use none but curvilinear coordinate systems in which the basis defined by (4) is orthogonal. The vectors of the basis of such a curvilinear coordinate system possess the following properties:

$$\begin{gathered}
(\mathbf{e}_1 \cdot \mathbf{e}_2) = (\mathbf{e}_1 \cdot \mathbf{e}_3) = (\mathbf{e}_2 \cdot \mathbf{e}_3) = 0 \\
\mathbf{e}_1 = [\mathbf{e}_2 \times \mathbf{e}_3], \quad \mathbf{e}_2 = [\mathbf{e}_3 \times \mathbf{e}_1], \quad \mathbf{e}_3 = [\mathbf{e}_1 \times \mathbf{e}_2]
\end{gathered} \tag{5}$$

The Cartesian basis $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ and the basis $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ are connected through formulas (2) and (4).

Let us consider the special case of cylindrical coordinates. Formula (1) takes the form

$$x = \rho \cos \varphi$$
$$y = \rho \sin \varphi$$
$$z = z$$

or, if we put $q_1 = \rho$, $q_2 = \varphi$, and $q_3 = z$, then

$$\mathbf{r} = \rho \cos \varphi \mathbf{i} + \rho \sin \varphi \mathbf{j} + z\mathbf{k}$$

We then calculate the Lamé parameters using formulas (3):

$$H_1 = \sqrt{\left(\frac{\partial x}{\partial \rho}\right)^2 + \left(\frac{\partial y}{\partial \rho}\right)^2 + \left(\frac{\partial z}{\partial \rho}\right)^2} = \sqrt{\cos^2 \varphi + \sin^2 \varphi} = 1$$
$$H_2 = \rho, \quad H_3 = 1$$

Formulas (4) give us the relation between the Cartesian and cylindrical bases (we denote the second by $\mathbf{e}_\rho$, $\mathbf{e}_\varphi$, $\mathbf{e}_z$):

$$\begin{aligned} \mathbf{e}_\rho &= \frac{\partial \mathbf{r}}{\partial \rho} = \cos \varphi \mathbf{i} + \sin \varphi \mathbf{j} \\ \mathbf{e}_\varphi &= \frac{1}{\rho} \frac{\partial \mathbf{r}}{\partial \varphi} = -\sin \varphi \mathbf{i} + \cos \varphi \mathbf{j} \\ \mathbf{e}_z &= \frac{\partial \mathbf{r}}{\partial z} = \mathbf{k} \end{aligned} \tag{6}$$

A direct check verifies the orthogonality of this basis, i.e. $(\mathbf{e}_\rho \cdot \mathbf{e}_\varphi) = (\mathbf{e}_\rho \cdot \mathbf{e}_z) = (\mathbf{e}_\varphi \cdot \mathbf{e}_z) = 0$.

We can easily find the inverse of (6):

$$\begin{aligned} \mathbf{i} &= -\mathbf{e}_\varphi \sin \varphi + \mathbf{e}_\rho \cos \varphi \\ \mathbf{j} &= \mathbf{e}_\varphi \cos \varphi + \mathbf{e}_\rho \sin \varphi \\ \mathbf{k} &= \mathbf{e}_z \end{aligned} \tag{7}$$

The projections of a vector $\mathbf{a}$ on a curvilinear basis, i.e.

$$\mathbf{a} = a_x \mathbf{i} + a_y \mathbf{j} + a_z \mathbf{k} = a_\rho \mathbf{e}_\rho + a_\varphi \mathbf{e}_\varphi + a_z \mathbf{e}_z \tag{8}$$

can be found if we substitute (6) into (8); thus

$$\begin{aligned} a_x &= a_\rho \cos \varphi - a_\varphi \sin \varphi \\ a_y &= a_\rho \sin \varphi + a_\varphi \cos \varphi \\ a_z &= a_z \end{aligned} \tag{9}$$

In a similar manner the spherical coordinate system

$$x = r \sin\theta \cos\varphi$$
$$y = r \sin\theta \sin\varphi$$
$$z = r \cos\theta$$

is related to the Cartesian system through the following formulas, which we give for purposes of reference:

$$\begin{aligned} &H_1 = 1,\ H_2 = r,\ H_3 = r\sin\theta \\ &\mathbf{e}_r = \sin\theta\cos\varphi\mathbf{i} + \sin\theta\sin\varphi\mathbf{j} + \cos\theta\mathbf{k} \\ &\mathbf{e}_\theta = \cos\theta\cos\varphi\mathbf{i} + \cos\theta\sin\varphi\mathbf{j} - \sin\theta\mathbf{k} \\ &\mathbf{e}_\varphi = -\sin\varphi\mathbf{i} + \cos\varphi\mathbf{j} \end{aligned} \tag{10}$$

It is easy to prove that this basis is orthogonal, i.e.

$$(\mathbf{e}_r \cdot \mathbf{e}_\varphi) = (\mathbf{e}_r \cdot \mathbf{e}_\theta) = (\mathbf{e}_\varphi \cdot \mathbf{e}_\theta) = 0$$

The components are related in the following manner:

$$\begin{aligned} a_x &= a_r \sin\theta\cos\varphi + a_\theta\cos\theta\cos\varphi - a_\varphi\sin\varphi \\ a_y &= a_r \sin\theta\sin\varphi + a_\theta\cos\theta\sin\varphi + a_\varphi\cos\varphi \\ a_z &= a_r\cos\theta - a_\theta\sin\theta \end{aligned} \tag{11}$$

When r and z are used simultaneously, we turn to the parabolic coordinates:

$$u = r + z, \quad v = r - z, \quad \varphi = \arctan\frac{y}{x}$$

The name *parabolic* originated from the fact that the surfaces u = constant and v = constant are paraboloids of revolution. (We can check this by squaring $r = u - z$ and $r = v + z$. We then obtain the following equalities:

$$x^2 + y^2 = u^2 - 2uz, \quad x^2 + y^2 = v^2 + 2vz$$

which are paraboloids of revolution if we put u and v constant.)

Putting $u = q_1$, $v = q_2$, $\varphi = q_3$, we get

$$x = \sqrt{q_1 q_2}\cos q_3, \quad y = \sqrt{q_1 q_2}\sin q_3, \quad z = (q_1 - q_2)/2$$

Calculating the Lamé parameters via formulas (3), we find, respectively,

$$H_1=\sqrt{\frac{v+u}{4u}},\quad H_2=\sqrt{\frac{v+u}{4v}},\quad H_3=\sqrt{uv}$$

3. Differential operators in curvilinear coordinates

grad f in orthogonal coordinate systems. In the Cartesian coordinate system grad f is determined in the following way:

$$\begin{aligned}\operatorname{grad} f&=\left(\frac{\partial f}{\partial x}\mathbf{i}+\frac{\partial f}{\partial y}\mathbf{j}+\frac{\partial f}{\partial z}\mathbf{k}\right)\\
&=\left(\frac{\partial f}{\partial q_1}\frac{\partial q_1}{\partial x}+\frac{\partial f}{\partial q_2}\frac{\partial q_2}{\partial x}+\frac{\partial f}{\partial q_3}\frac{\partial q_3}{\partial x}\right)\mathbf{i}\\
&\qquad+\left(\frac{\partial f}{\partial q_1}+\frac{\partial q_1}{\partial y}\frac{\partial f}{\partial q_2}\frac{\partial q_2}{\partial y}+\frac{\partial f}{\partial q_3}\frac{\partial q_3}{\partial y}\right)\mathbf{j}\\
&\qquad+\left(\frac{\partial f}{\partial q_1}\frac{\partial q_1}{\partial z}+\frac{\partial f}{\partial q_2}\frac{\partial q_2}{\partial z}+\frac{\partial f}{\partial q_3}\frac{\partial q_3}{\partial z}\right)\mathbf{k}\\
&=\frac{\partial f}{\partial q_1}\left(\frac{\partial q_1}{\partial x}\mathbf{i}+\frac{\partial q_1}{\partial y}\mathbf{j}+\frac{\partial q_1}{\partial z}\mathbf{k}\right)\\
&\qquad+\frac{\partial f}{\partial q_2}\left(\frac{\partial q_2}{\partial x}\mathbf{i}+\frac{\partial q_2}{\partial y}\mathbf{j}+\frac{\partial q_2}{\partial z}\mathbf{k}\right)\\
&\qquad+\frac{\partial f}{\partial q_3}\left(\frac{\partial q_3}{\partial x}\mathbf{i}+\frac{\partial q^3}{\partial y}\mathbf{j}+\frac{\partial q_3}{\partial z}\mathbf{k}\right)\\
&=\frac{\partial f}{\partial q_1}\operatorname{grad} q_1+\frac{\partial f}{\partial q_2}\operatorname{grad} q_2+\frac{\partial f}{\partial q_3}\operatorname{grad} q_3 \qquad (1)\end{aligned}$$

Let us consider the vector grad q_1. The scalar products of this vector by $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ are, respectively,

$$(\operatorname{grad} q_1\cdot\mathbf{e}_1)=\frac{1}{H_1}\left(\frac{\partial q_1}{\partial x}\frac{\partial x}{\partial q_1}+\frac{\partial q_1}{\partial y}\frac{\partial y}{\partial q_1}+\frac{\partial q_1}{\partial z}\frac{\partial z}{\partial q_1}\right)=\frac{1}{H_1}$$

$$(\operatorname{grad} q_1\cdot\mathbf{e}_2)=\frac{1}{H_2}\left(\frac{\partial q_1}{\partial x}\frac{\partial x}{\partial q_2}+\frac{\partial q_1}{\partial y}\frac{\partial y}{\partial q_2}+\frac{\partial q_1}{\partial z}\frac{\partial z}{\partial q_2}\right)=0$$

$$(\operatorname{grad} q_1\cdot\mathbf{e}_3)=0$$

This suggests that grad q_1 is directed along $\mathbf{e}_1$ and that its absolute value equals $1/H_1$:

$$\operatorname{grad} q_1=\frac{1}{H_1}\mathbf{e}_1 \qquad (2)$$

In a similar manner

$$\operatorname{grad} q_2 = \frac{1}{H_2}\,\mathbf{e}_2$$

$$\operatorname{grad} q_3 = \frac{1}{H_3}\,\mathbf{e}_3$$

Substituting the formulas obtained into (1), we get

$$\operatorname{grad} f = \frac{1}{H_1}\frac{\partial f}{\partial q_1}\,\mathbf{e}_1 + \frac{1}{H_2}\frac{\partial f}{\partial q_2}\,\mathbf{e}_2 + \frac{1}{H_3}\frac{\partial f}{\partial q_3}\,\mathbf{e}_3 \tag{3}$$

For cylindrical coordinates

$$H_1 = 1,\ H_2 = \rho,\ H_3 = 1$$

$$\operatorname{grad} f = \frac{\partial f}{\partial \rho}\,\mathbf{e}_\rho + \frac{1}{\rho}\frac{\partial f}{\partial \varphi}\,\mathbf{e}_\varphi + \frac{\partial f}{\partial z}\,\mathbf{e}_z \tag{4}$$

For spherical coordinates

$$H_1 = 1,\ H_2 = r,\ H_3 = r\sin\theta$$

$$\operatorname{grad} f = \frac{\partial f}{\partial r}\,\mathbf{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\,\mathbf{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \varphi}\,\mathbf{e}_\varphi \tag{5}$$

div a in orthogonal coordinate systems. In the Cartesian coordinate system div **a** is defined in the following way:

$$\operatorname{div}\mathbf{a} = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z}$$

If we expand **a** along the basis vectors of a curvilinear coordinate system, we can write [see formula (14) in Appendix 1]:

$$\operatorname{div}(a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3) = a_1 \operatorname{div}\mathbf{e}_1 + a_2\operatorname{div}\mathbf{e}_2 + a_3\operatorname{div}\mathbf{e}_3 + (\mathbf{e}_1\cdot\operatorname{grad} a_1) + (\mathbf{e}_2\cdot\operatorname{grad} a_2) + (\mathbf{e}_3\cdot\operatorname{grad} a_3) \tag{6}$$

To complete the calculation we must find div $\mathbf{e}_1$, div $\mathbf{e}_2$, and div $\mathbf{e}_3$. For this we take the curl of (2):

$$\operatorname{curl}\operatorname{grad} q_1 = \operatorname{curl}\left(\frac{1}{H_1}\,\mathbf{e}_1\right) = 0 \tag{7}$$

Making use of formulas in Appendix 1, we obtain

$$\frac{1}{H_1}\operatorname{curl}\mathbf{e}_1 - \frac{1}{H_1^2}\,[\operatorname{grad} H_1 \times \mathbf{e}_1] = 0$$

whence

$$\operatorname{curl} \mathbf{e}_1 = \frac{1}{H_1} [\operatorname{grad} H_1 \times \mathbf{e}_1] \tag{8}$$

According to formula (3)

$$\operatorname{grad} H_1 = \frac{1}{H_1} \frac{\partial H_1}{\partial q_1} \mathbf{e}_1 + \frac{1}{H_2} \frac{\partial H_1}{\partial q_2} \mathbf{e}_2 + \frac{1}{H_3} \frac{\partial H_1}{\partial q_3} \mathbf{e}_3 \tag{9}$$

Substituting (9) into (8) and making use of formula (5), Appendix 2, we obtain

$$\operatorname{curl} \mathbf{e}_1 = \frac{1}{H_1 H_3} \frac{\partial H_1}{\partial q_3} \mathbf{e}_2 - \frac{1}{H_1 H_2} \frac{\partial H_1}{\partial q_2} \mathbf{e}_3 \tag{10}$$

In a similar manner

$$\operatorname{curl} \mathbf{e}_2 = \frac{1}{H_2 H_1} \frac{\partial H_2}{\partial q_1} \mathbf{e}_3 - \frac{1}{H_3 H_2} \frac{\partial H_2}{\partial q_3} \mathbf{e}_1 \tag{11}$$

$$\operatorname{curl} \mathbf{e}_3 = \frac{1}{H_3 H_2} \frac{\partial H_3}{\partial q_2} \mathbf{e}_1 - \frac{1}{H_3 H_1} \frac{\partial H_3}{\partial q_1} \mathbf{e}_3 \tag{12}$$

Since by formula (5) of Appendix 2 $\mathbf{e}_1 = [\mathbf{e}_2 \times \mathbf{e}_3]$, it follows from formula (16), Appendix 1, that

$$\operatorname{div} \mathbf{e}_1 = \operatorname{div} [\mathbf{e}_2 \times \mathbf{e}_3] = \mathbf{e}_3 \operatorname{curl} \mathbf{e}_2 - \mathbf{e}_2 \operatorname{curl} \mathbf{e}_3 \tag{13}$$

Substituting (11) and (12) into (13), we obtain

$$\operatorname{div} \mathbf{e}_1 = \frac{1}{H_1 H_2} \frac{\partial H_2}{\partial q_1} + \frac{1}{H_1 H_3} \frac{\partial H_3}{\partial q_1} \tag{14}$$

In a similar manner

$$\operatorname{div} \mathbf{e}_2 = \frac{1}{H_2 H_3} \frac{\partial H_3}{\partial q_2} + \frac{1}{H_1 H_2} \frac{\partial H_1}{\partial q_2} \tag{15}$$

$$\operatorname{div} \mathbf{e}_3 = \frac{1}{H_1 H_3} \frac{\partial H_1}{\partial q_3} + \frac{1}{H_2 H_3} \frac{\partial H_2}{\partial q_3} \tag{16}$$

Substituting (14)-(16) and grad a_1, grad a_2 and grad a_3 [using formula (3)] into (6) and carrying out the required transformations, we get

$$\operatorname{div} \mathbf{a} = \frac{1}{H_1 H_2 H_3} \left[\frac{\partial}{\partial q_1} (a_1 H_2 H_3) + \frac{\partial}{\partial q_2} (a_2 H_1 H_3) - \frac{\partial}{\partial q_3} (a_3 H_1 H_2) \right] \tag{17}$$

By substituting the Lamé parameters of the cylindrical coordinate system into (17) we get the final formula for

div **a**:

$$\operatorname{div}\mathbf{a}=\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho a_\rho)+\frac{1}{\rho}\frac{\partial a_\varphi}{\partial\varphi}+\frac{\partial a_z}{\partial z}\tag{18}$$

In the case of spherical coordinates

$$\operatorname{div}\mathbf{a}=\frac{1}{r^2}\frac{\partial}{\partial r}(r^2a_r)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta a_\theta)+\frac{1}{r^2\sin^2\theta}\frac{\partial a_\varphi}{\partial\varphi}\tag{19}$$

curl a in orthogonal coordinate systems.

$$\begin{aligned}\operatorname{curl}\mathbf{a}&=\operatorname{curl}(a_1\mathbf{e}_1+a_2\mathbf{e}_2+a_3\mathbf{e}_3)=a_1\operatorname{curl}\mathbf{e}_1\\&+a_2\operatorname{curl}\mathbf{e}_2+a_3\operatorname{curl}\mathbf{e}_3+[\operatorname{grad}a_1\times\mathbf{e}_1]\\&+[\operatorname{grad}a_2\times\mathbf{e}_2]+[\operatorname{grad}a_3\times\mathbf{e}_3]\end{aligned}\tag{20}$$

Substituting (10)-(12) and grad a_1, grad a_2 and grad a_3 [using formula (3)] into (20) and carrying out the required transformations, we get

$$\begin{aligned}\operatorname{curl}\mathbf{a}&=\frac{1}{H_2H_3}\left[\frac{\partial}{\partial q_2}(a_3H_3)-\frac{\partial}{\partial q_3}(a_2H_2)\right]\mathbf{e}_1\\&+\frac{1}{H_1H_3}\left[\frac{\partial}{\partial q_3}(a_1H_1)-\frac{\partial}{\partial q_1}(a_3H_3)\right]\mathbf{e}_2\\&+\frac{1}{H_1H_2}\left[\frac{\partial}{\partial q_1}(a_2H_2)-\frac{\partial}{\partial q_2}(a_1H_1)\right]\mathbf{e}_3\end{aligned}\tag{21}$$

For purposes of reference we give the formula for curl **a** in cylindrical coordinates:

$$\begin{aligned}\operatorname{curl}\mathbf{a}&=\left(\frac{1}{\rho}\frac{\partial a_z}{\partial\varphi}-\frac{\partial a_\varphi}{\partial z}\right)\mathbf{e}_\rho+\left(\frac{\partial a_\rho}{\partial z}-\frac{\partial a_z}{\partial\rho}\right)\mathbf{e}_\varphi\\&+\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho a_\varphi)-\frac{1}{\rho}\frac{\partial a_\rho}{\partial\varphi}\right)\mathbf{e}_z\end{aligned}\tag{22}$$

In the case of spherical coordinates

$$\begin{aligned}\operatorname{curl}\mathbf{a}&=\frac{1}{r\sin\theta}\left\{\frac{\partial}{\partial\theta}(\sin\theta a_\varphi)-\frac{\partial a_\theta}{\partial\varphi}\right\}\mathbf{e}_r\\&+\frac{1}{r}\left\{\frac{1}{\sin\theta}\frac{\partial a_r}{\partial\varphi}-\frac{\partial}{\partial r}(ra_\varphi)\right\}\mathbf{e}_\theta+\frac{1}{r}\left\{\frac{\partial}{\partial r}(ra_\theta)-\frac{\partial a_r}{\partial\theta}\right\}\mathbf{e}_\varphi\end{aligned}\tag{23}$$

Δf in orthogonal coordinate systems. Since $\Delta f = \operatorname{div}\operatorname{grad} f$, it follows that after substituting grad f [see

formula (3)] into (17) we obtain

$$\operatorname{div}\operatorname{grad} f = \Delta f = \frac{1}{H_1 H_2 H_3}\left[\frac{\partial\left(\frac{H_2 H_3}{H_1}\frac{\partial f}{\partial q_1}\right)}{\partial q_1} + \frac{\partial\left(\frac{H_3 H_1}{H_2}\frac{\partial f}{\partial q_2}\right)}{\partial q_2} + \frac{\partial\left(\frac{H_1 H_2}{H_3}\frac{\partial f}{\partial q_3}\right)}{\partial q_3}\right]$$

In the particular case of cylindrical coordinates

$$\Delta f = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial f}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial\varphi^2} + \frac{\partial^2 f}{\partial z^2} \tag{24}$$

in spherical coordinates

$$\Delta f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin\theta}\frac{\partial^2 f}{\partial\varphi^2} \tag{25}$$

in parabolic coordinates

$$\Delta f = \frac{4}{u+v}\left\{\frac{\partial}{\partial u}\left(u\frac{\partial f}{\partial u}\right) + \frac{\partial}{\partial v}\left(v\frac{\partial f}{\partial v}\right) + \frac{u+v}{4uv}\frac{\partial^2 f}{\partial\varphi^2}\right\} \tag{26}$$

$\Delta\mathbf{a}$ in orthogonal coordinate systems. The explicit form of

$$\Delta\mathbf{a} = \frac{\partial^2\mathbf{a}}{\partial x^2} + \frac{\partial^2\mathbf{a}}{\partial y^2} + \frac{\partial^2\mathbf{a}}{\partial z^2}$$

in curvilinear coordinates is found with the help of formula (12), Appendix 1:

$$\Delta\mathbf{a} = \operatorname{grad}\operatorname{div}\mathbf{a} - \operatorname{curl}\operatorname{curl}\mathbf{a}$$

Since this expression in the general case is cumbersome, we write out $\Delta\mathbf{a}$ for purposes of reference in cylindrical coordinates

$$\Delta\mathbf{a} = \left(\Delta a_\rho - \frac{a_\rho}{\rho^2} - \frac{2}{\rho^2}\frac{\partial a_\varphi}{\partial\varphi}\right)\mathbf{e}_\rho + \frac{1}{2}\left(\Delta a_\varphi - \frac{a_\varphi}{\rho^2} + \frac{2}{\rho^2}\frac{\partial a_\rho}{\partial\varphi}\right)\mathbf{e}_\varphi + \Delta a_z\mathbf{e}_z; \tag{27}$$

in spherical coordinates

$$\Delta \mathbf{a} = \left[\Delta a_r - \frac{2}{r^2} \left(a_r + \frac{1}{\sin\theta} \frac{\partial}{\partial\theta} (\sin\theta a_\theta) + \frac{1}{\sin\theta} \frac{\partial a_\varphi}{\partial\varphi} \right) \right] \mathbf{e}_r$$
$$+ \left[\Delta a_\theta + \frac{2}{r^2} \left(\frac{\partial a_r}{\partial\theta} - \frac{a_\theta}{2\sin^2\theta} - \frac{\cos\theta}{\sin^2\theta} \frac{\partial a_\varphi}{\partial\varphi} \right) \right] \mathbf{e}_\theta$$
$$+ \left[\Delta a_\varphi + \frac{2}{r^2 \sin\theta} \left(\frac{\partial a_r}{\partial\varphi} + \cot\theta \frac{\partial a_\theta}{\partial\varphi} - \frac{a_\varphi}{2\sin\theta} \right) \right] \mathbf{e}_\varphi \quad (28)$$

In formulas (27) and (28) the operator Δ acts on scalar functions, as in (24) and (25) respectively.

4. Mathematical supplement

The Dirac delta function. Let $f(x)$ and all its derivatives be continuous functions in the interval $(-\infty, \infty)$. The delta function is then defined by the relationship

$$\int_{-\infty}^{\infty} f(x)\, \delta(x-a)\, dx = f(a) \quad (1)$$

where $f(x) \to 0$ as $x \to \pm\infty$.

It follows from the definition that

(i) $\int_{-\infty}^{\infty} \delta(x)\, dx = 1,$

(ii) $\delta(\alpha x) = \frac{1}{|\alpha|} \delta(x),$

(iii) $\delta^n(x) = (-1)^n \delta(x),$

(iv) $\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx}\, dk$ (the Fourier expansion),

(v) $\delta[\varphi(x)] = \sum_{s=1}^{k} \frac{\delta(x-x_s)}{|\varphi'(x_s)|}$, where x_s are the simple roots of $\varphi(x) = 0$.

The gamma and (complete) beta functions. These are defined by the following integral equalities:

$$\Gamma(\alpha)=\int_0^\infty e^{-x}x^{\alpha-1}\,dx \quad (\alpha>0) \tag{2}$$

$$\mathrm{B}(\alpha,\beta)=\int_0^1 (1-x)^{\beta-1}x^{\alpha-1}\,dx=\frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)} \tag{3}$$

If we integrate (2) by parts, we get

$$\Gamma(\alpha+1)=\alpha\Gamma(\alpha) \tag{4}$$

For $\alpha = 1$ or $\alpha = 1/2$ the relationship (4) gives

$$\Gamma(1)=1 \tag{5}$$

$$\Gamma(1/2)=2\int_0^\infty e^{-y^2}\,dy=\sqrt{\pi} \tag{6}$$

Using these relationships, we can determine $\Gamma(\alpha)$ for $\alpha = \frac{n+1}{2}$, where $n = 0, 1, 2, \ldots$. For other values of α we must use special tables.

The volume of a sphere of radius R in n-dimensional space. The volume bounded by a surface $x_1^2+x_2^2+\ldots+x_n^2=R^2$ (the equation of a sphere in n-dimensional space) is

$$V_n(R)=\int\ldots\int dx_1\ldots dx_n \tag{7}$$

If we change the variables, using $x_i = y_iR$, we get

$$V_n(R)=R^nV_n(1) \tag{8}$$

where

$$V_n(1)=\int\ldots\int dy_1\ldots dy_n \tag{9}$$

Furthermore, the integration is carried over the domain limited by the surface $y_1^2+y_2^2+\ldots+y_n^2\leqslant 1$.

Using the relationships (3), (4), (9), we can easily calculate $V_n(1)$:

$$V_n(1) = \int_{-1}^{1} dy_1 \int_{y_2^2+\ldots+y_n^2 \leq 1-y_1^2} \!\!\!\ldots \int dy_2 \ldots dy_n$$

$$= \int_{-1}^{1} (1-y_1^2)^{\frac{n-1}{2}} dy_1 V_{n-1}(1)$$

$$= V_{n-1}(1)\, \mathrm{B}\left(\frac{1}{2}, \frac{n+1}{2}\right)$$

$$V_n(1) = \frac{\Gamma(1/2)\,\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)} V_{n-1}(1) \tag{10}$$

Now with this recurrence relation we get

$$V_n(1) = \frac{\Gamma(1/2)\,\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)} \frac{\Gamma(1/2)\,\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)} \ldots V_1(1) = \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)}$$

since $V_1(1) = 1$. Whence

$$V_n(R) = R^n \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)} \tag{11}$$

Calculation of integrals of type

$$I(\alpha) = \int_{-\infty}^{\infty} x^m e^{-\alpha x^n} dx \tag{12}$$

$$m = 0, 1, 2 \ldots$$
$$n = 2, 4, 6 \ldots$$

If m is odd, $I(\alpha)$ is zero. If m is even,

$$I(\alpha) = 2 \int_{0}^{\infty} x^m e^{-\alpha x^n} dx$$

The integral in the last expression reduces to the gamma function if we change variables, using $y = \alpha x^n$:

$$\int_{-\infty}^{\infty} x^m e^{-\alpha x^n}\, dx = \frac{2}{n} \frac{\Gamma\left(\frac{m+1}{n}\right)}{\alpha^{\frac{m+1}{n}}} \tag{13}$$

We note that the equality

$$\int_{0}^{\infty} x^m e^{-\alpha x^n}\, dx = \frac{1}{n} \frac{\Gamma\left(\frac{m+1}{n}\right)}{\alpha^{\frac{m+1}{n}}} \tag{14}$$

holds for all $m \geqslant 0$ and $n \geqslant 0$, and m and n are not required to be integers.

For particular cases expression (14) yields

(i) $$\int_{0}^{\infty} x e^{-\alpha x^2}\, dx = \frac{1}{2\alpha},$$

(ii) $$\int_{0}^{\infty} x^2 e^{-\alpha x^2}\, dx = \frac{\pi^{1/2}}{4\alpha^{3/2}},$$

(iii) $$\int_{0}^{\infty} x^3 e^{-\alpha x^2}\, dx = \frac{1}{2} \alpha^2,$$

(iv) $$\int_{0}^{\infty} x^4 e^{-\alpha x^2}\, dx = \frac{1}{4} \frac{\Gamma(3/2)}{\alpha^{3/2}}.$$

Error integral. The error integral is defined as

$$\Phi(x) = \operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_{0}^{x} e^{-t^2}\, dt \tag{15}$$

When $x \ll 1$, the integrand in (15) can be expanded in a power series; thus,

$$\Phi_{x \to 0}(x) = \frac{2}{\sqrt{\pi}} \left(x - \frac{x^3}{3 \times 1!} + \frac{x^5}{5 \times 2!} - \dots \right) \tag{16}$$

If we apply definition (15), we can show that

$$\left.\begin{aligned}\frac{2}{\sqrt{\pi}}\int_{\pm x}^{\infty} e^{-t^2}\,dt &= 1 \mp \Phi(x)\\ \frac{2}{\sqrt{\pi}}\int_{\pm x}^{\infty} t^2 e^{-t^2}\,dt &= \frac{1}{2} \pm \left\{\frac{x}{\sqrt{\pi}}e^{-x^2}-\frac{1}{2}\Phi(x)\right\}\end{aligned}\right\} \quad (17)$$

$$\Phi(1)=\frac{2}{\sqrt{\pi}}\int_0^1 e^{-t^2}\,dt \approx 0.84 \quad (18)$$

Some integrals of quantum statistics. In evaluating physical properties of quantum fermion systems we often meet an integral of the following type:

$$I_1(\mu)=\int_0^\infty \frac{f(\varepsilon)\,d\varepsilon}{\exp\left(\frac{\varepsilon-\mu}{kT}\right)+1}$$

where $f(\varepsilon)$ is such that the integral converges.

If we change the variables, using $y=\frac{\varepsilon-\mu}{kT}$, and restrict ourselves to low temperatures, we reduce the integral to the asymptotic series

$$I_1(\mu)=kT\int_0^{\mu/kT}\frac{f(\mu-kTy)}{e^{-y}+1}\,dy+kT\int_0^\infty\frac{f(\mu+kTy)}{e^y+1}\,dy$$

$$=kT\int_0^\infty\frac{f(\mu+kTy)-f(\mu-kTy)}{e^y+1}\,dy+\int_0^\mu f(\varepsilon)\,d\varepsilon$$

Hence

$$I_1(\mu)=\int_0^\mu f(\varepsilon)\,d\varepsilon+2kT\sum_{n=1}^{\infty}\frac{(kT)^{2n-1}}{(2n-1)}f^{(2n-1)}(\mu)\int_0^\infty\frac{y^{2n-1}}{e^y+1}\,dy \quad (19)$$

But

$$\int_0^\infty \frac{y^{2n-1}}{e^y+1}\,dy = \int_0^\infty y^{2n-1}e^{-y}\sum_{m=0}^{\infty}(-1)^{me-my}\,dy$$

$$= \Gamma(2n)\sum_{m=0}^{\infty}\frac{(-1)^m}{(m+1)^{2n}}$$

$$= \Gamma(2n)\,(1-2^{1-2n})\,\zeta(2n)$$

where $\zeta(2n)=\sum_{m=1}^{\infty} m^{-2n}$ is the Riemann zeta function, which for some values of n is

$$\zeta(3/2)=2.612,\quad \zeta(5/2)=1.341,\quad \zeta(2)=\frac{\pi^2}{6},$$

$$\zeta(4)=\frac{\pi^4}{90},\qquad \zeta(3)=1.202,\quad \zeta(5)=1.037$$

And so,

$$I_1(\mu)=\int_0^\mu f(\varepsilon)\,d\varepsilon$$

$$+2kT\sum_{n=1}^{\infty}\frac{(kT)^{2n-1}}{(2n-1)!}f^{(2n-1)}(\mu)\,\Gamma(2n)\,[1-2^{1-2n}]\,\zeta(2n) \quad (20)$$

In evaluating physical properties of quantum boson systems we often meet an integral of another type:

$$I_2(\mu)=\int_0^\infty \frac{y^{n-1}}{e^y-1}\,dy$$

Calculations similar to the previous one yield

$$I_2(\mu)=\int_0^\infty \frac{y^{n-1}}{e^y-1}\,dy=\Gamma(n)\,\zeta(n) \quad (21)$$

Returning to $I_1(\mu)$, we must note that if $f(\varepsilon)=\frac{\varepsilon^j}{\Gamma(j+1)}$, the integral is called the Fermi integral:

$$F_j(\eta)\equiv\frac{1}{\Gamma(j+1)}\int_0^\infty \frac{\varepsilon^j\,d\varepsilon}{\exp(\varepsilon-\eta)+1} \quad (22)$$

Since the Fermi integral is widely used in the theory of solids, it has been tabulated for a great number of values of j. In the simplest case of $j = 1/2$ we give estimates for $F_j(\eta)$:

$$F_{1/2}(\eta) \approx e^{\eta} \quad \text{as } \eta \to -\infty \tag{23}$$

$$F_{1/2}(\eta) \approx \frac{e^{\eta}}{1+0.27e^{\eta}} \quad \text{for} \quad \eta \leqslant 1.3$$

$$F_{1/2}(\eta) \approx \frac{4\eta^{3/2}}{3\sqrt{\pi}}\left(1+\frac{1.15}{\eta^2}\right) \quad \text{for} \quad \eta \geqslant 1 \tag{24}$$

The condition for the totality of the differential of a function of two variables. We call the linear part of the increment of a function $f = f(x, y)$ the total differential, i.e.

$$df = \left(\frac{\partial f}{\partial x}\right)_y dx + \left(\frac{\partial f}{\partial y}\right)_x dy \equiv f_x\, dx + f_y\, dy$$

The necessary and sufficient condition for the expression

$$dF(x, y) = Q(x, y)\, dx + P(x, y)\, dy$$

to be a total differential when Q, P, and their derivatives, $\left(\frac{\partial Q}{\partial y}\right)_x$ and $\left(\frac{\partial P}{\partial x}\right)_y$, exist and are continuous is the condition that

$$\left(\frac{\partial Q}{\partial y}\right)_x = \left(\frac{\partial P}{\partial x}\right)_y \tag{25}$$

If condition (25) holds,

$$\oint dF = 0$$

Functional determinants of two functions and their properties. The functional determinant or Jacobian of two functions $u(x, y) = u$ and $v(x, y) = v$ is the determinant

$$\frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \left(\frac{\partial u}{\partial x}\right)_y & \left(\frac{\partial v}{\partial x}\right)_y \\ \left(\frac{\partial u}{\partial y}\right)_x & \left(\frac{\partial v}{\partial y}\right)_x \end{vmatrix} \tag{26}$$

If we use this definition and the properties of determinants, we can prove the following relationships for Jacobians:

$$\frac{\partial (u, v)}{\partial (x, v)} = \left(\frac{\partial u}{\partial x}\right)_v \tag{27}$$

$$\frac{\partial (u, v)}{\partial (v, x)} = -\left(\frac{\partial u}{\partial x}\right)_v \tag{28}$$

$$\frac{\partial (u, v)}{\partial (x, y)} = \frac{\partial (u, v)}{\partial (p, q)} \times \frac{\partial (p, q)}{\partial (x, y)} \tag{29}$$

If $u = u(p, q, l)$, $v = v(p, q, l)$, and if $p = p(x, y)$, $q = q(x, y)$, $l = l(x, y)$, then

$$\frac{\partial (u, v)}{\partial (x, y)} = \frac{\partial (u, v)}{\partial (p, q)} \times \frac{\partial (p, q)}{\partial (x, y)} + \frac{\partial (u, v)}{\partial (q, l)} \frac{\partial (q, l)}{\partial (x, y)} + \frac{\partial (u, v)}{\partial (l, p)} \times \frac{\partial (l, p)}{\partial (x, y)} \tag{30}$$

5. Legendre polynomials

The equation

$$\frac{d}{dx}\left[(1-x^2)\frac{dP_l(x)}{dx}\right] + l(l+1)P_l(x) = 0 \tag{1}$$

has for its solution the Legendre polynomials

$$P_l(x) = \frac{1}{2^l l!}\frac{d^l (x^2-1)^l}{dx^l} \tag{2}$$

We can see this if for $Z = (x^2 - 1)^l$ we compose the identity $(x - 1)\frac{dZ}{dx} - 2lxZ = 0$ and differentiate it l times with respect to x.

According to equality (2)

$$P_0(x) = 1, \qquad P_1(x) = x,$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1), \quad P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

etc., and we see that $P_l(-x) = (-1)^l P_l(x)$.

The function $\frac{1}{|\mathbf{r}' - \mathbf{r}|}$ is the generating function for $P_l(x)$ when $r' = 1$ and $r < 1$. Denoting the angle between $\mathbf{r}$ and $\mathbf{r}'$ by arc cos x, we write

$$F(r, x) = \frac{1}{\sqrt{1 - 2rx + r^2}} = \sum_{l=0}^{\infty} P_l(x)\, r^l \tag{3}$$

It is evident that for $r > 1$ by introducing the variable $r_1 = 1/r$ we get the expansion

$$\frac{1}{\sqrt{1-2rx+r^2}} = \sum_{l=0}^{\infty} P_l(x)\, r^{-l-1}$$

$P_l(x)$ can also be defined in the following way:

$$P_l(x) = \frac{1}{2\pi i} \int\limits_L \frac{1}{2^l} \frac{(z^2-1)^l}{(-x+z)^{l+1}}\, dz \tag{4}$$

where L is an arbitrarily chosen closed contour encompassing the point $z = x$, which is the pole of the integrand. Expanding $f(z) = (z^2 - 1)^l$ into a series in powers of $z - x$ and using the theory of residues, we see that definition (4) is consistent with definition (2). Substituting $\zeta/2$ for $\frac{z-x}{z^2-1}$ in (4), with $z = \frac{1}{\zeta}\,(1 - \sqrt{1 - 2x\zeta + \zeta^2})$ (we choose z's that turn into x as $\zeta \to 0$), we get

$$P_l(x) = \frac{1}{2\pi i} \frac{1}{l!} \oint \frac{d\zeta}{\zeta^{l+1} \sqrt{1-2\zeta x+\zeta^2}}$$
$$= \frac{1}{l!} \left[\frac{\partial^l}{\partial x^l} \frac{1}{\sqrt{1-2\zeta x+\zeta^2}} \right]_{\zeta=0}$$

This demonstrates the equivalence of (3) and (4).

Using formula (3), we can easily obtain the recurrence relations between Legendre polynomials and their derivatives.

By composing the product $(1 - 2xr + r^2)\,\frac{\partial F}{\partial r}$, we get the equality

$$\frac{x-r}{\sqrt{1-2xr+r^2}} = \sum_{l=0}^{\infty} (x-r)\, P_l(x)\, r^l$$
$$= \sum_{l=0}^{\infty} (1-2xr+r^2)\, l P_l(x)\, r^{l-1} \tag{5}$$

Identifying powers of r, we find the first recurrence relation:

$$(l+1)\, P_{l+1}(x) - (2l+1)\, x P_l(x) + l P_{l-1}(x) = 0 \tag{6}$$

We compose the sum $F+2r\frac{\partial F}{\partial r}$ and get

$$F+2r\frac{\partial F}{\partial r}=\frac{1-r^2}{(\sqrt{1-2xr+r^2})^3}=\sum_{l=0}^{\infty}(2l+1)P_l(x)r^l \qquad (7)$$

On the other hand

$$r^2\frac{\partial F}{\partial r}+rF=\frac{r-xr^2}{(\sqrt{1-2xr+r^2})^3}=\sum_{l=0}^{\infty}lP_{l-1}(x)r^l \qquad (8)$$

which, if added to (5), yields

$$\frac{x-r}{(\sqrt{1-2xr+r^2})^3}+\frac{r-xr^2}{(\sqrt{1-2xr+r^2})^3}$$

$$=\frac{x(1-r^2)}{(\sqrt{1-2xr+r^2})^3}=\sum_{l=0}^{\infty}lP_{l-1}r^l+\sum_{l=0}^{\infty}lP_lr^{l-1}$$

Comparing the last expression with (7), we get

$$(2l+1)\,xP_l(x)=(l+1)\,P_{l+1}(x)+lP_{l-1}(x)$$

If we compose the product $(1-2xr+r^2)\frac{\partial F}{\partial x}$, we find the second recurrence relation:

$$P_l(x)=P'_{l+1}(x)-2xP'_l(x)+P'_{l-1}(x) \qquad (9)$$

Substituting P'_{l+1} from (6) into (9), multiplying the obtained expression $P_l(x)=xP'_l(x)-P'_{l-1}(x)$ by 2, and adding it to (9), we find the third recurrence relation:

$$(2l+1)\,P_l(x)=\frac{dP_{l+1}(x)}{dx}-\frac{dP_{l-1}(x)}{dx} \qquad (10)$$

We can find the normalization integral for Legendre polynomials if we use definition (2). Let us suppose that $l \geqslant k$.

Then

$$\int_{-1}^{1} P_k(x) P_l(x)\, dx = \frac{1}{2^l l!\, 2^k k!} \int_{-1}^{1} \frac{d^l (x^2-1)^l}{dx^l} \frac{d^k (x^2-1)^k}{dx^k}\, dx = \dots$$

$$= \frac{1}{2^l l!\, 2^k k!} (-1)^l \int_{-1}^{1} (x^2-1)^l \frac{d^{l+k} (x^2-1)^k}{dx^{k+l}}\, dx$$

$$= \frac{2}{2l+1} \delta_{kl}$$

where we integrate by parts, bearing in mind that on the boundaries $x = \pm 1$ the function $(x^2 - 1)$ turns zero.

6. Hermite polynomials

Hermite polynomials $H_n(x)$ can be defined on the interval $-\infty \leqslant x \leqslant \infty$ by means of the generating function

$$F(x, t) = e^{-t^2+2tx} = e^{x^2} e^{-(t-x)^2} = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n \tag{1}$$

from which we immediately get

$$H_n(x) = \left(\frac{\partial^n F(x, t)}{\partial x^n} \right)_{t=0} = (-1)^n e^{x^2} \frac{d^n (e^{-x^2})}{dx^n} \tag{2}$$

The equality $\frac{\partial F(x, t)}{\partial x} = 2t \frac{\partial F(x, t)}{\partial t}$ with the use of (1) takes the form

$$\sum_{n=0}^{\infty} \frac{1}{n!} H_n'(x) t^n = 2 \sum_{n=0}^{\infty} t \frac{H_n(x)}{n!} t^n$$

whence, by identifying powers of t, we get the relation

$$\frac{dH_n(x)}{dx} = 2nH_{n-1}(x) \tag{3}$$

In a similar manner from the equality $\frac{\partial F(x, t)}{\partial t} + 2(t - x) F(x, t) = 0$ we obtain the relation

$$H_{n+1}(x) - 2xH_n(x) + 2nH_{n-1}(x) = 0 \quad (n \geqslant 1) \tag{4}$$

Combining (3) and (4), we come to the homogeneous linear equation for $H_n(x)$

$$H_n''(x) - 2xH_n'(x) + 2nH_n(x) = 0 \qquad (5)$$

The orthogonality of Hermite polynomials can be found from the following relation:

$$\int_{-\infty}^{\infty} H_m(x) H_n(x) e^{-x^2} dx = (-1)^n \int_{-\infty}^{\infty} H_m(x) \frac{d^n (e^{-x^2})}{dx^n} dx$$

where we use formula (2) and set $n > m$. Integrating by parts and noting that for $x = \pm\infty$ all the derivatives of e^{-x^2} turn zero, we obtain

$$\int_{-\infty}^{\infty} H_m(x) H_n(x) e^{-x^2} dx = (-1)^{n-1} 2m \int_{-\infty}^{\infty} H_{m-1}(x) \frac{d^{n-1} e^{-x^2}}{dx^{n-1}}$$

$$= \ldots = (-1)^{n-m} 2^m \times m! \int_{-\infty}^{\infty} H_0(x) \frac{d^{n-m} (e^{-x^2})}{dx^{n-m}} dx = 0$$

where we have used formula (3). To normalize $H_n(x)$ we put n equal to m. Then

$$\int_{-\infty}^{\infty} H_n^2(x) e^{-x^2} dx = 2^n \times n! \sqrt{\pi}$$

The orthonormalized functions are the functions

$$\varphi_n(x) = \frac{1}{(2^n \times n! \sqrt{\pi})^{1/2}} H_n(x) e^{-x^2}$$

$$(n = 0, 1, 2, \ldots)$$

The general expression for $H_n(x)$

$$H_n(x) = (2x)^n - \frac{n(n-1)}{1!}(2x)^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2!}(2x)^{n-4} - \ldots$$

makes it possible to compute any one of them. For instance,

$$H_0(x) = 1, \qquad H_1(x) = 2x,$$

$$H_2(x) = 4x^2 - 2, \quad H_3(x) = 8x^3 - 12x$$

In addition $H_n(-x) = (-1)^n H_n(x)$.

7. The confluent hypergeometric function

For all z and a finite and for any value of c not equal to 0, -1, -2, ... this function is defined by the series

$$F(a, c; z) = 1 + \frac{a}{c}\frac{z}{1!} + \frac{a(a+1)}{c(c+1)}\frac{z^2}{2!} + \ldots (F(a, a; z) = e^z) \quad (1)$$

The confluent hypergeometric function (often called Kummer's confluent hypergeometric function) is a particular solution of the equation

$$z\frac{d^2\Phi}{dz^2} + (c-z)\frac{d\Phi}{dz} - a\Phi = 0 \quad (2)$$

The second solution of (2) is the confluent hypergeometric function of the second kind

$$z^{1-c}F(a-c+1,\ 2-c;\ z) \quad (3)$$

whenever c is not an integer. The function $F(a, c; z)$ satisfies a number of relations, which follow from formulas (1) and (2):

$$F(a, c; z) = e^z F(c-a, c; z)$$

$$(c-a)F(a-1, c; z) + (2a-c+z)F(a, z; z)$$
$$= aF(a+1, c; z) \quad (4)$$

$$\frac{d}{dz}F(a, c; z) = \frac{a}{c}F(a+1, c+1; z)$$

$$\frac{d^n}{dz^n}F(a, c; z) = \frac{\Gamma(c)}{\Gamma(a)} \times \frac{\Gamma(a+n)}{\Gamma(c+n)} \times F(a+n, c+n; z)$$

If a is a negative integer, $a = -n$, or zero, the function $F(a, c; z)$ reduces to a polynomial of degree n:

$$F(-n, c; z) = 1 - \frac{n}{c}z + \ldots + (-1)^n\frac{(c-1)!}{(c+n-1)!}z^n$$
$$= \frac{z^{1-c}e^z\Gamma(c)}{\Gamma(c+n)}\frac{d^n}{dz^n}(z^{c+n-1} \times e^{-z}) \quad (5)$$

The asymptotics of $F(a; c; z)$ are

$$\operatorname{Re} z \to \infty\colon\ F(a, c; z) = \frac{\Gamma(c)}{\Gamma(a)} z^{a-c} \cdot e^{z} \cdot [1 + O(|z|^{-1})]$$

$$\operatorname{Re} z \to -\infty\colon\ F(a, c; z) = \frac{\Gamma(c)}{\Gamma(c-a)} (-z)^{-a} [1 + O(|z|^{-1})] \qquad (6)$$

$\operatorname{Re} c \to \infty$: $F(a, c; z) = 1 + O(|c|^{-1})$ for finite z and a

Any equation of type

$$(a_0 x + b_0)\frac{d^2\varphi}{dx^2} + (a_1 x + b_1)\frac{d\varphi}{dx} + (a_2 x + b_2)\varphi = 0 \qquad (7)$$

can be reduced to (2) by the following substitutions:

$$\varphi = e^{\nu x}\Phi, \quad x = \lambda z + \mu$$

where ν, λ, μ are determined from the system of equations

$$\begin{gathered} a_0\mu + b_0 = 0 \\ a_0 + \lambda(2a_0\nu + a_1) = 0 \\ a_0\nu^2 + a_1\nu + a_2 = 0 \end{gathered} \qquad (8)$$

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